ON DIRICHLET'S CONJECTURE ON RELATIVE CLASS NUMBER ONE

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Abstract. We examine relative class numbers, associated to class numbers of quadratic fields $\mathbb{Q}(\sqrt{m})$ for $m > 0$ and square-free. The relative class number is the quotient:

$$H_d(f) = \frac{h(f^2 d)}{h(d)},$$

where d is the discriminant of $\mathbb{Q}(\sqrt{m})$ and h refers to the class number. It is not known if for every m there exists an $f > 1$ for which this ratio is one, although Dirichlet conjectured that this is true. We prove that there does exist such an f when $\sqrt{m}$ has a particular continued fraction form. The main result concerns when the continued fraction is diagonal, i.e. when all entries are equal.

1. Introduction

Compared to imaginary quadratic fields when $m < 0$, very little is known about the class number problem for real quadratic fields. Properties of the relative class number for $m > 0$ are even more elusive. An open question in this area is whether there is a relative class number of 1 in every real quadratic field. Dirichlet conjectured that this is true and in this paper we narrow down the possibilities of where it may not hold true by finding a relative class number of 1 for certain values of m. The continued fraction expansions of $\sqrt{m}$ follows specific patterns that enable us to guarantee relative class numbers of one for many values of m at once. We use a similar proof for each case although they rapidly become more complicated as the period length of the continued fractions lengthen. We prove Dirichlet's conjecture for continued fraction expansions with period lengths of 1, 2 and 3 as well as all cases where $\sqrt{m} = \langle n, \overline{a, a, \ldots, a, 2n} \rangle$. In addition, we prove the conjecture for some special cases of period lengths 4 and 5.

Sections 2, 3, and 4 will provide the necessary background for Dirichlet's formula for computing the relative class number, which is introduced in Section 5. Section 6 will then give background on continued fractions which will lead in to our research and results in Sections 7 and 8.

2. Quadratic Fields

The space we will be working in is a real quadratic field $\mathbb{Q}(\sqrt{m})$ for m a positive square-free integer (i.e. $p^2 \nmid m$ for any prime p). Any element in $\mathbb{Q}(\sqrt{m})$ can be written in the form $x + y\sqrt{m}$ where $x, y \in \mathbb{Q}$ and every number that can be written in this form is an element of $\mathbb{Q}(\sqrt{m})$.

We will now introduce some characteristics associated with quadratic fields and special elements of $\mathbb{Q}(\sqrt{m})$ and standardize notation.

Definition 2.1. The *norm* of an element in $\mathbb{Q}(\sqrt{m})$ is defined to be $N(x + y\sqrt{m}) = x^2 - y^2 m$.

Definition 2.2. The *field discriminant* d for a quadratic field $\mathbb{Q}(\sqrt{m})$ is defined as follows:

$$d = \begin{cases} m & m \equiv 1 \pmod 4 \\ 4m & m \equiv 2, 3 \pmod 4 \end{cases}$$

Definition 2.3. A *quadratic integer* in $\mathbb{Q}(\sqrt{m})$ is any root of a monic quadratic polynomial with integer coefficients, $t^2 + bt + c$ where m is the square-free part of $b^2 - 4c$ (i.e. $b^2 - 4c = f^2 \cdot m$ for some $f \in \mathbb{Z}$). The set of all quadratic integers is then $\{\frac{x + y\sqrt{m}}{z} \mid x, y \in \mathbb{Z} \text{ and } x \equiv y \pmod z\}$ where

$$z = \begin{cases} 2 & m \equiv 1 \pmod 4 \\ 1 & m \equiv 2, 3 \pmod 4 \end{cases}$$

These integers form a ring, denoted $\mathcal{O}$. Note that the integers depend on m, but this notation assumes that the value of m is understood in context.

Definition 2.4. In a given quadratic field and for a given integer $f > 0$, define $\mathcal{O}_f$ as the quadratic integers for which the y-term is divisible by f. That is, $\mathcal{O}_f = \{w = \frac{x + y\sqrt{m}}{z} \mid w \in \mathcal{O}, y \equiv 0 \pmod f, \text{ and } z \text{ is defined as above } \}$

Definition 2.5. A *unit* in $\mathcal{O}$ is any invertible quadratic integer.

The following proposition will help to characterize the units in $\mathcal{O}$.

Since quadratic integers have norms in $\mathbb{Z}$, it follows that the units in $\mathcal{O}$ are those quadratic integers that have norm ± 1. For $m > 0$, these units form an infinite cyclic group. The smallest positive generator of this group we call the *fundamental unit*, denoted ε_m. We will now explore this in the following section.

3. Fundamental Unit

Let m be a square-free positive integer. Because units are quadratic integers with norm ± 1, ε_m is equal to $\frac{x+y\sqrt{m}}{z}$ where z is defined as above and (x,y) is the smallest positive solution to the Pell equation $x^2 - y^2 m = \pm z^2$. That is,

$$\varepsilon_m = \begin{cases} x + y\sqrt{m} & \text{where } (x,y) \text{ is the smallest positive solution to} \\ & x^2 - y^2 m = \pm 1 \text{ when } m \equiv 2,3 \pmod 4 \\ \dfrac{x+y\sqrt{m}}{2} & \text{where } (x,y) \text{ is the smallest positive solution to} \\ & x^2 - y^2 m = \pm 4 \text{ when } m \equiv 1 \pmod 4 \end{cases}$$

We have an algorithm for solving these Pell equations.

Algorithm 3.1. [?] To solve $x^2 - y^2 m = \pm z^2$, first let $r_0 = 0$, $q_0 = 1$, and $a_0 = \lfloor \sqrt{m} \rfloor$. Starting with $i = 0$, update r, q, and a as follows:

$$r_{i+1} = a_i \cdot q_i - r_i$$

$$q_{i+1} = \frac{m - r_{i+1}^2}{q_i}$$

$$a_{i+1} = \lfloor \frac{a_0 + r_{i+1}}{q_{i+1}} \rfloor$$

incrementing i by one until $q_{i+1} = 1$ or z^2.

Now, let $h_{-2} = 0$, $h_{-1} = 1$, $k_{-2} = 1$, $k_{-1} = 0$. For $j = 0, 1, \ldots, i$:

$$h_j = a_j \cdot h_{j-1} + h_{j-2}$$

$$k_j = a_j \cdot k_{j-1} + k_{j-2}$$

This will give $h_i^2 - k_i^2 d = (-1)^{i+1} q_{i+1}$.

Example 3.2. Let $m = 61$. Because $m \equiv 1 \pmod 4$, we want to solve $x^2 - y^2 \cdot 61 = \pm 4$. We go through the algorithm until $q = 4$ or 1.

i	0	1	2	3	4
r_i	0	7	5	7	
q_i	1	12	3	4	
a_i	7	1	4	3	

Because $q_3 = 4$, we now want to find h_2 and k_2:

i	-2	-1	0	1	2
a_i			7	1	4
h_i	0	1	7	8	39
k_i	1	0	1	1	5

This gives us the equation $39^2 - 5^2 \cdot 61 = -4$, so $(x, y) = (39, 5)$. Thus the fundamental unit for $d = 61$ is $\varepsilon_{61} = \dfrac{39 + 5\sqrt{61}}{2}$.

4. Quadratic Reciprocity

We now introduce the Legendre symbol along with some of its useful properties, which will appear in our main theorem in Section 5.

Definition 4.1. The symbol $\left(\dfrac{a}{p}\right)$ is called the *Legendre symbol* and is defined as follows:

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \mid a \\ 1 & \text{if } x^2 \equiv a \pmod{p} \text{ has a solution} \\ -1 & \text{if } x^2 \equiv a \pmod{p} \text{ has no solution} \end{cases}$$

where $a, p \in \mathbb{Z}$ and p prime.

The following theorem presents properties that help us determine the value of the Legendre symbol in difficult cases.

Theorem 4.2. [?] *Let $a, b, p \in \mathbb{Z}$. Then the following properties hold:*

(1) *If $a \equiv b \pmod{p}$ then* $\left(\dfrac{a}{p}\right) = \left(\dfrac{b}{p}\right)$

(2) $\left(\dfrac{ab}{p}\right) = \left(\dfrac{a}{p}\right)\left(\dfrac{b}{p}\right)$

(3) *If p, q are odd primes with $p \neq q$, then*

$$\left(\frac{q}{p}\right) = (-1)^{\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)}\left(\frac{p}{q}\right)$$

5. The Class Number and Relative Class Number

Now that we have provided the appropriate background, we define the main concept of the paper, the class number $h(f)$, and a variation of it, the relative class number $H_d(f)$.

Define an equivalence relation $\sim$ on all ideals in $\mathbb{Q}(\sqrt{m})$:

$$I \sim J \text{ iff } \exists\ \rho, \sigma \in \mathcal{O} \text{ such that } \rho I = \sigma J$$

Then the number of equivalence classes that this relation partitions $\mathbb{Q}(\sqrt{m})$ into is the class number, denoted $h(d)$.

An alternative to studying $h(d)$ is to look at the relative class number, defined to be the ratio

$$H_d(f) = \frac{h(f^2 d)}{h(d)}$$

where $h(f^2d)$ refers to the class number definition above but with $\rho, \sigma \in \mathcal{O}_f$. It should be clear that $h(d) \leq h(f^2d)$ since $\mathcal{O}_f \subseteq \mathcal{O}$. Rather than computing this ratio directly, we have a formula to compute the relative class number shown in the following theorem from Dirichlet.

Theorem 5.1 (Dirichlet)**.** [?] *Let m be a fixed, square-free, positive integer, and d be the field discriminant of $\mathbb{Q}(\sqrt{m})$. Define $\psi(f) = f \prod_{q|f} \left(1 - \left(\frac{d}{q}\right)\frac{1}{q}\right)$ where $\left(\frac{d}{q}\right)$ is the Legendre symbol and q is prime. Define $\phi(f)$ to be the smallest positive integer such that $(\varepsilon_m)^{\phi(f)} \in \mathcal{O}_f$, i.e. $(\varepsilon_m)^{\phi(f)} = \frac{x + y\sqrt{m}}{z}$ where $y \equiv 0 \ (mod\ f)$. Then*

$$H_d(f) = \frac{\psi(f)}{\phi(f)}$$

Dirichlet conjectured that for every m and corresponding d there exists an f such that $H_d(f) = 1$, although this remains an open question. We examine this problem by looking at the continued fraction expansions of $\sqrt{m}$ for certain values of m, which will be preceded by some background on continued fractions.

6. Continued Fractions

We begin by defining the type of continued fraction we are interested in.

Definition 6.1. The *infinite periodic continued fraction* denoted $\langle a_0, \ldots, a_n, \overline{b_0, \ldots, b_m} \rangle$ is equal to $\langle a_0, \ldots, a_n, b_0, \ldots, b_m, b_0, \ldots, b_m, b_0, \ldots \rangle$

$$= a_0 + \cfrac{1}{\ldots + \cfrac{1}{a_n + \cfrac{1}{b_0 + \cfrac{1}{\ldots + \cfrac{1}{b_m + \cfrac{1}{b_0 + \ldots}}}}}}$$

More specifically, $\sqrt{m}$ has a certain form of infinite periodic continued fraction expansion.

Theorem 6.2. [?] *The continued fraction expansion of $\sqrt{m}$ for a positive integer m that is not a perfect square is $\langle n, \overline{a_1, a_2, \ldots, a_2, a_1, 2n} \rangle$ where $n = \lfloor \sqrt{m} \rfloor$.*

Definition 6.3. [?] For any continued fraction $\langle a_0, a_1, a_2, \ldots \rangle$, a *convergent* $\frac{h_i}{k_i} = \frac{a_i h_{i-1} + h_{i-2}}{a_i k_{i-1} + k_{i-2}}$ where $i \geq 0$ and $h_{-2} = 0$, $h_{-1} = 1$, $k_{-2} = 1$, and $k_{-1} = 0$.

7. Main Results

Our main result is the proof of a relative class number of one for all m values such that $\sqrt{m} = \left\langle n, \overline{a, a, \ldots, a, 2n} \right\rangle$. We lead up to this theorem with three simple cases: $\left\langle n, \overline{2n} \right\rangle$, $\left\langle n, \overline{a, 2n} \right\rangle$, and $\left\langle n, \overline{a, a, 2n} \right\rangle$. We approach each of these simple cases by finding a general form of m, using our algorithm to find the fundamental unit, then proving the existence of an f that gives a relative class number of one.

7.1. The Base Case $\left\langle n, \overline{2n} \right\rangle$.

We will examine the continued fraction expansions of $\sqrt{m}$, beginning with the simplest case $\left\langle n, \overline{2n} \right\rangle$. We start by solving the continued fraction for a general form of m.

Lemma 7.1. *The continued fraction* $\left\langle n, \overline{2n} \right\rangle = n + \dfrac{1}{2n + \frac{1}{2n + \ldots}}$*, is equal to* $\sqrt{n^2 + 1}$.

Proof. Let $x = \left\langle \overline{2n} \right\rangle$. So $\left\langle n, \overline{2n} \right\rangle = n + x^{-1}$. Observe that $x = \left\langle \overline{2n} \right\rangle = \left\langle 2n, \overline{2n} \right\rangle = \langle 2n, x \rangle$. So

$$
\begin{aligned}
x &= 2n + \frac{1}{x} \\
x^2 &= 2nx + 1 \\
x^{-2} + 2nx^{-1} - 1 &= 0 \\
x^{-1} &= \frac{-2n \pm \sqrt{4n^2 + 4}}{2} \\
&= -n \pm \sqrt{n^2 + 1}
\end{aligned}
$$

Because we are only considering positive values of m, we disregard the negative solution to get $\left\langle n, \overline{2n} \right\rangle = n + x^{-1} = n + (-n + \sqrt{n^2 + 1}) = \sqrt{n^2 + 1}$. $\square$

Now we will find a general form for the fundamental unit ε_m in order to use it in our definition of $\phi(f)$.

Lemma 7.2. *Let* $\sqrt{m} = \left\langle n, \overline{2n} \right\rangle = \sqrt{n^2 + 1}$. *Then the fundamental unit is* $\varepsilon_m = n + \sqrt{m}$.

Proof. We can use our previous algorithm to find the fundamental unit of this general value of m.

i	0	1	2
r_i	0	n	
q_i	1	1	
a_i	n	$2n$	

Because $q_1 = 1$, we now want to find h_0 and k_0:

i	-2	-1	0	1
a_i			n	$2n$
h_i	0	1	n	
k_i	1	0	1	

This gives us the equation $n^2 - 1^2 \cdot m = -1$. So if $m \equiv 2, 3 \pmod 4$, the fundamental unit for $m = n^2 + 1$ is $\varepsilon = n + \sqrt{m}$. If $m \equiv 1 \pmod 4$, we get $(2n)^2 - (2 \cdot 1)^2 \cdot m = -4$ so the fundamental unit is $\varepsilon = \dfrac{2n + 2\sqrt{m}}{2} = n + \sqrt{m}$. □

Finally, we outline how to find an f that will give us $H_d(f) = 1$.

Theorem 7.3. *Let $m = n^2 + 1$ and f be any prime such that $f \mid m$. Then $H_d(f) = \dfrac{\psi(f)}{\phi(f)} = 1$.*

Proof. Since f is prime, $\psi(f) = f \prod_{q \mid f} \left(1 - \left(\frac{d}{q}\right)\frac{1}{q}\right) = f\left(1 - \left(\frac{d}{f}\right)\frac{1}{f}\right)$. Because $f \mid m$, $f \mid d$ also and so the Legendre symbol $\left(\frac{d}{f}\right) = 0$. This gives us $\psi(f) = f$. Now, we know that $\dfrac{\psi(f)}{\phi(f)} \in \mathbb{Z}$, so $\phi(f) = 1$ or f. By definition, $\phi(f)$ is the smallest positive integer such that $f \mid y$ where $\varepsilon^{\phi(f)} = x + y\sqrt{m}$. By Theorem **??**, the fundamental unit $\varepsilon = n + \sqrt{m}$ so since $f \nmid 1$, $\phi(f) \neq 1$. Thus, $\phi(f) = f$ and $H_d(f) = 1$.

□

We have now shown that for m values with $\sqrt{m} = \langle n, \overline{2n} \rangle$ as their continued fraction expansion (i.e. m values that are 1 greater than a perfect square), there must exist an f that gives us $H_d(f) = 1$.

7.2. **The $\langle n, \overline{a, 2n} \rangle$ Case.** The next case is $\langle n, \overline{a, 2n} \rangle$ and will be very similar to the first case. We will again start by solving the continued fraction for a general form of m.

Lemma 7.4. *The continued fraction $\langle n, \overline{a, 2n} \rangle = n + \cfrac{1}{a + \cfrac{1}{2n + \cfrac{1}{a + \cfrac{1}{2n + \dots}}}}$, is equal to $\sqrt{n^2 + \dfrac{2n}{a}}$ where $a \mid 2n$ and $n^2 + \dfrac{2n}{a}$ is not a perfect square.*

Proof. Let $x = \langle \overline{a, 2n} \rangle$. So $\langle n, \overline{a, 2n} \rangle = n + x^{-1}$. Observe that $x = \langle \overline{a, 2n} \rangle = \langle a, 2n, \overline{a, 2n} \rangle = \langle a, 2n, x \rangle$. So

$$x = a + \frac{1}{2n + x^{-1}}$$

$$\begin{aligned} 2nx + 1 &= 2an + ax^{-1} + 1 \\ 2n &= 2anx^{-1} + ax^{-2} \\ ax^{-2} + 2anx^{-1} - 2n &= 0 \\ x^{-1} &= \frac{-2an \pm \sqrt{4a^2n^2 + 4(2an)}}{2a} \\ &= -n \pm \sqrt{n^2 + \frac{2n}{a}} \end{aligned}$$

Disregarding the negative solution gives us $\langle n, \overline{a, 2n} \rangle = n + x^{-1} = n + \left(-n + \sqrt{n^2 + \frac{2n}{a}}\right) = \sqrt{n^2 + \frac{2n}{a}}$. Since $m \in \mathbb{Z}$ and m is square-free, we must have that $a \mid 2n$ and $n^2 + \frac{2n}{a}$ is not a perfect square.

□

Next we find the general form of the fundamental unit.

Lemma 7.5. *Let* $\sqrt{m} = \langle n, \overline{a, 2n} \rangle = \sqrt{n^2 + \frac{2n}{a}}$. *Then the fundamental unit is* $\varepsilon_m = (an + 1) + a\sqrt{m}$.

Proof. Expanding our table from the previous case gives us

i	0	1	2	3
r_i	0	n	n	
q_i	1	$\frac{2n}{a}$	1	
a_i	n	a	$2n$	

Because $q_2 = 1$, we now want to find h_1 and k_1:

i	-2	-1	0	1	2
a_i			n	a	$2n$
h_i	0	1	n	$an + 1$	
k_i	1	0	1	a	

This gives us the equation $(an + 1)^2 - a^2 \cdot m = 1$. So the fundamental unit for $m = n^2 + \frac{2n}{a}$ is $\varepsilon_m = (an + 1) + a\sqrt{m}$ □

It should be noted that the a row in our algorithm for finding the fundamental unit is exactly the sequence of numbers in the continued fraction expansion for $\sqrt{m}$. This will be useful in the remaining cases, as we will not have to go through the first part of the algorithm; we may jump ahead to finding h_i and k_i from the row of a values.

We now prove the existence of an f that will give us a relative class number of 1.

Lemma 7.6. *Let* $m = n^2 + \dfrac{2n}{a}$. *Then there exists a prime* f *such that* $f \mid m$ *and* $f \nmid a$.

Proof. We know that $\dfrac{2n}{a} > 1$ since if $\dfrac{2n}{a} = 1$, we would be in the $\langle n, \overline{2n} \rangle$ case. So $a < 2n$.

Now, if $n = 1$, we must have $a = 1$ and $m = 3$. On the other hand, if $n \geq 2$ we have $a < 2n \leq n^2 < m$. So in both cases we have $a < m$ and since m is square-free, we are guaranteed the existence of a prime that divides m and does not divide a.

□

Theorem 7.7. *Let* $m = n^2 + \dfrac{2n}{a}$ *and* f *be any prime such that* $f \mid m$ *and* $f \nmid a$. *Then* $H_d(f) = \dfrac{\psi(f)}{\phi(f)} = 1$.

Proof. Since f is prime, $\psi(f) = f \prod_{q \mid f} \left(1 - \left(\dfrac{d}{q}\right)\dfrac{1}{q}\right) = f\left(1 - \left(\dfrac{d}{f}\right)\dfrac{1}{f}\right)$. Because $f \mid m$, $f \mid d$ also and so $\left(\dfrac{d}{f}\right) = 0$ and $\psi(f) = f$. Now, we know that $\dfrac{\psi(f)}{\phi(f)} \in \mathbb{Z}$ so $\phi(f) = 1$ or f. By definition, $\phi(f)$ is the smallest positive integer such that $f \mid y$ where $\varepsilon^{\phi(f)} = x + y\sqrt{m}$. By Theorem **??** the fundamental unit $\varepsilon = (an+1) + a\sqrt{m}$ so since $f \nmid a$, $\phi(f) \neq 1$. Thus, $\phi(f) = f$ and $H_d(f) = 1$. □

In this case, we chose an f that did not divide a because it turned out the a was the y-term of the fundamental unit. It will always be the case that the denominator in the fraction of the general form of m will be the y-term of the fundamental unit. So by choosing such an f we are always guaranteed to get a relative class number of 1.

7.3. The $\langle n, \overline{a, a, 2n} \rangle$ Case.

Once again, we start by solving the continued fraction expansion for m.

Lemma 7.8. *The continued fraction* $\langle n, \overline{a, a, 2n} \rangle = n + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{2n + \cfrac{1}{a + \dots}}}}$, *is equal to* $\sqrt{n^2 + \dfrac{2an+1}{a^2+1}}$ *where* $a^2 + 1 \mid 2an + 1$ *and* $n^2 + \dfrac{2an+1}{a^2+1}$ *is not a perfect square.*

Proof. Let $x = \langle\overline{a,a,2n}\rangle$. So $\langle n,\overline{a,a,2n}\rangle = n + x^{-1}$. Note that $x = \langle\overline{a,a,2n}\rangle = \langle a,a,2n,x\rangle$. So

$$x = a + \cfrac{1}{a + \frac{1}{2n+x^{-1}}}$$

$$x = a + \cfrac{1}{\frac{2an+ax^{-1}+1}{2n+x^{-1}}}$$

$$x = a + \frac{2n+x^{-1}}{2an+ax^{-1}+1}$$

$$2anx + a + x = 2a^2n + a^2x^{-1} + a + 2n + x^{-1}$$

$$(a^2+1)x^{-2} + (2a^2n+2n)x^{-1} - (2an+1) = 0$$

$$\begin{aligned} x^{-1} &= \frac{-2a^2n - 2n \pm \sqrt{(2a^2n+2n)^2 + 4(a^2+1)(2an+1)}}{2(a^2+1)} \\ &= -n \pm \frac{1}{2(a^2+1)}\sqrt{n^2(2a^2+2) + 2(2a^2+2)(2an+1)} \\ &= -n \pm \sqrt{n^2 + \frac{2an+1}{a^2+1}} \end{aligned}$$

Disregarding the negative solution gives us $\langle n,\overline{a,a,2n}\rangle = n + x^{-1} = n + \left(-n + \sqrt{n^2 + \frac{2an+1}{a^2+1}}\right) = \sqrt{n^2 + \frac{2an+1}{a^2+1}}$.

Again, $m \in \mathbb{Z}$ and m is square-free, so $a^2+1 \mid 2an+1$ and $n^2 + \frac{2an+1}{a^2+1}$ is not a perfect square.

$\square$

Next we find the fundamental unit for this particular m.

Lemma 7.9. *Let* $\sqrt{m} = \langle n,\overline{a,a,2n}\rangle = \sqrt{n^2 + \frac{2an+1}{a^2+1}}$. *Then the fundamental unit is* $\varepsilon_m = (a^2n + a + n) + (a^2+1)\sqrt{m}$.

Proof. As stated earlier, we may begin the fundamental unit algorithm at the second stage, using the sequence in the continued fraction expansion as our a row.

i	-2	-1	0	1	2	3
a_i			n	a	a	$2n$
h_i	0	1	n	$an+1$	a^2n+a+n	
k_i	1	0	1	a	a^2+1	

This gives us the equation $(a^2n + a + n)^2 - (a^2 + 1)^2 \cdot m = -1$ so the fundamental unit for $m = n^2 + \dfrac{2an+1}{a^2+1}$ is $\varepsilon_m = (a^2n + a + n) + (a^2+1)\sqrt{m}$. □

We now provide the criteria for an f that will give us a relative class number of 1, which follows the pattern introduced in the first two cases.

Lemma 7.10. *Let $m = n^2 + \dfrac{2an+1}{a^2+1}$. Then a is even and there exists a prime f such that $f \mid m$ and $f \nmid a^2 + 1$.*

Proof. Let $k = \dfrac{2an+1}{a^2+1}$. If a were odd, then the denominator of this fraction would be even. But the numerator is odd, and this would make k not an integer. This is a contradiction since we know k must be an integer, so a must be even.

Now, since $k \in \mathbb{Z}$,

$$\begin{aligned} 2an+1 &\equiv 0 \pmod{a^2+1} \\ 2an &\equiv -1 \pmod{a^2+1} \\ 2an &\equiv a^2 \pmod{a^2+1} \\ 2n &\equiv a \pmod{a^2+1} \\ n &\equiv \frac{a}{2} \pmod{a^2+1} \end{aligned}$$

So $n = l(a^2+1) + \dfrac{a}{2}$ for some positive integer l (because if $l = 0$, we are in the $\langle n, \overline{2n}\rangle$ case) and $m = n^2 + k = \left(l(a^2+1) + \dfrac{a}{2}\right)^2 + k > a^2 + 1$. Thus, there must be a prime f that divides m and does not divide $a^2 + 1$. □

Theorem 7.11. *Let $m = n^2 + \dfrac{2an+1}{a^2+1}$ and f be a prime such that $f \mid m$ and $f \nmid a^2 + 1$. Then $H_d(f) = \dfrac{\psi(f)}{\phi(f)} = 1$.*

The proof is omitted here simply because it is nearly identical to the proofs of the corresponding theorems in the previous cases.

We have now shown that for all m values with the property that $\sqrt{m} = \langle n, \overline{a, a, 2n}\rangle$, there exists an f such that $H_d(f) = 1$.

7.4. **The $\langle n, \overline{a, a, \ldots, a, 2n}\rangle$ Case.** We will now examine the cases where $\sqrt{m} = \langle n, \overline{a, a, \ldots, a, 2n}\rangle = \sqrt{n^2 + \dfrac{P_r}{Q_r}}$ where r represents the number of

a's in the continued fraction expansion of $\sqrt{m}$. Recall from the previous cases that $P_1 = 2n$, $Q_1 = a$, $P_2 = 2an + 1$, and $Q_2 = a^2 + 1$.

Since we cannot directly solve for a general form of m, we will instead look at some properties of P_r and Q_r and use induction to find a general form of n that will lead to a relative class number of one.

The first two properties define how to find P_r and Q_r from previous r values, as shown in the following lemma.

Lemma 7.12. *Let* $\sqrt{m} = \left\langle n, \overline{a, a, \ldots, a, 2n} \right\rangle = \sqrt{n^2 + \frac{P_r}{Q_r}}$ *and* $x = \left\langle \overline{a, a, \ldots, a, 2n} \right\rangle = \langle a, a, \ldots, a, 2n, x \rangle$. *Then* $P_r = aP_{r-1} + P_{r-2}$ *and* $Q_r = aQ_{r-1} + Q_{r-2}$ *for all integers* $r > 2$.

Proof. By Theorem **??**, the convergents of $x = \langle a, a, \ldots, a, 2n, x \rangle$ are

$$\frac{h_0}{k_0} = \frac{a}{1} = a$$

$$\frac{h_1}{k_1} = \frac{a^2+1}{a}$$

$$\frac{h_2}{k_2} = \frac{a^3+2a}{a^2+1}$$

$$\vdots$$

$$\frac{h_r}{k_r} = \frac{2n \cdot h_{r-1} + h_{r-2}}{2n \cdot k_{r-1} + k_{r-2}}$$

$$\frac{h_{r+1}}{k_{r+1}} = \frac{x \cdot h_r + h_{r-1}}{x \cdot k_r + k_{r-1}}$$

Note that for $i < r$, $k_i = h_{i-1}$, $h_i = ah_{i-1} + h_{i-2}$ and $k_i = ak_{i-1} + k_{i-2}$, while $h_r = 2nh_{r-1} + h_{r-2}$ and $k_r = 2nk_{r-1} + k_{r-2}$.

Since there are $r+2$ entries in the continued fraction of x, the convergent $\frac{h_{r+1}}{k_{r+1}}$ is equal to x. So now we have

$$k_r \cdot x^2 + k_{r-1} \cdot x = h_r \cdot x + h_{r-1}$$

$$h_{r-1} \cdot x^{-2} + (h_r - k_{r-1})x^{-1} - k_r = 0$$

Note that $h_r - k_{r-1} = 2n \cdot h_{r-1} + h_{r-2} - k_{r-1} = 2n \cdot h_{r-1}$.

Solving for x^{-1} gives us

$$x^{-1} = \frac{-(2n \cdot h_{r-1}) + \sqrt{(2n \cdot h_{r-1})^2 + 4h_{r-1} \cdot k_r}}{2h_{r-1}}$$

$$= -n + \sqrt{n^2 + \frac{k_r}{h_{r-1}}}$$

So

$$\begin{aligned}\frac{P_r}{Q_r} &= \frac{k_r}{h_{r-1}} = \frac{2n\cdot k_{r-1}+k_{r-2}}{h_{r-1}}\\ &= \frac{2n(ak_{r-2}+k_{r-3})+ak_{r-3}+k_{r-4}}{ah_{r-2}+h_{r-3}}\\ &= \frac{a(2n\cdot k_{r-2}+k_{r-3})+2n\cdot k_{r-3}+k_{r-r}}{ah_{r-2}+h_{r-3}}\\ &= \frac{a\cdot P_{r-1}+P_{r-2}}{a\cdot Q_{r-1}+Q_{r-2}}\end{aligned}$$

Thus $P_r = aP_{r-1} + P_{r-2}$ and $Q_r = aQ_{r-1} + Q_{r-2}$ for all integers $r > 2$.

□

The next lemma shows how P_r and Q_r are related to each other.

Lemma 7.13. *Let* $\sqrt{m} = \langle n, \overline{a, a, \ldots, a, 2n}\rangle = \sqrt{n^2 + \frac{P_r}{Q_r}}$. *Then* $P_r = 2nQ_{r-1} + Q_r - aQ_{r-1}$.

Proof. We have shown in previous cases that $P_1 = 2n$, $Q_1 = a$, $P_2 = 2an+1$, and $Q_2 = a^2+1$. So the formula holds when $r = 2$ since

$$\begin{aligned}2nQ_1 + Q_2 - aQ_1 &= 2an + a^2 + 1 - a^2\\ &= 2an+1\\ &= P_2\end{aligned}$$

Now assume $P_r = 2nQ_{r-1} + Q_r - aQ_{r-1}$ for all $r < R$ for some integer $R > 2$. Then, using our formula from the previous theorem,

$$\begin{aligned}P_R &= aP_{R-1} + P_{R-2}\\ &= a(2nQ_{R-2} + Q_{R-1} - aQ_{R-2}) + P_{R-2}\\ &= 2anQ_{R-2} + aQ_{R-1} - a^2Q_{R-2} + 2nQ_{R-3} + Q_{R-2} - aQ_{R-3}\\ &= 2n(aQ_{R-2} + Q_{R-3}) + aQ_{R-1} + Q_{R-2} - a(aQ_{R-2} - Q_{R-3})\\ &= 2nQ_{R-1} + Q_R - aQ_{R-1}\end{aligned}$$

Thus the formula holds for all integers $r > 1$.

□

The next lemma defines the simple relationship between n and a for a given r by using the previous lemma.

Lemma 7.14. *Let* $\sqrt{m} = \langle n, \overline{a, a, \ldots, a, 2n}\rangle = \sqrt{n^2 + \frac{P_r}{Q_r}}$. *Then* $2n \equiv a \pmod{Q_r}$.

Proof. We know that since $\dfrac{P_r}{Q_r} \in \mathbb{Z}$, $P_r \equiv 0 \pmod{Q_r}$. By reducing the formula in the previous theorem modulo Q_r we get

$$0 \equiv 2nQ_{r-1} - aQ_{r-1} \pmod{Q_r}.$$

So $2nQ_{r-1} \equiv aQ_{r-1} \pmod{Q_r}$ and thus $2n \equiv a \pmod{Q_r}$. □

With this relationship between n and a, we can now find a general form of n that will guarantee the existence of a prime that will give us a relative class number of one.

Lemma 7.15. *Let* $\sqrt{m} = \langle n, \overline{a, a, \ldots, a, 2n} \rangle = \sqrt{n^2 + \dfrac{P_r}{Q_r}}$. *Then there exists a prime* f *such that* $f \mid m$ *and* $f \nmid Q_r$.

Proof. By the previous lemma, $2n \equiv a \pmod{Q_r}$. So we have three possibilities: a and Q_r are both even, a and Q_r are both odd, or a is even and Q_r is odd. These give us the following values of n:

$$n = \begin{cases} \dfrac{a}{2} + l \cdot \dfrac{Q_r}{2}, \text{ for some } l > 0 & \text{when } a \text{ and } Q_r \text{ are both even} \\ \dfrac{a + Q_r}{2} + l \cdot Q_r, \text{ for some } l \geq 0 & \text{when } a \text{ and } Q_r \text{ are both odd} \\ \dfrac{a}{2} + l \cdot Q_r, \text{ for some } l > 0 & \text{when } a \text{ is even and } Q_r \text{ is odd} \end{cases}$$

In each of these cases, we get $m = n^2 + \dfrac{P_r}{Q_r} > Q_r$. Thus there must exist a prime that divides m and does not divide Q_r. □

Theorem 7.16. *Let* $\sqrt{m} = \langle n, \overline{a, a, \ldots, a, 2n} \rangle = \sqrt{n^2 + \dfrac{P_r}{Q_r}}$ *and let* f *be any prime such that* $f \mid m$ *and* $f \nmid Q_r$. *Then* $H_d(f) = \dfrac{\psi(f)}{\phi(f)} = 1$.

Proof. Just as in the proofs of the previous cases, because f is prime and a divisor of m, $\psi(f) = f$. Since Q_r is the y-term of the fundamental unit and $f \nmid Q_r$, $\phi(f) \neq 1$. Thus we must have $\phi(f) = f$ and $H_d(f) = 1$. □

8. Miscellaneous Results

We now move on to the more general $\langle n, \overline{a, b, a, 2n} \rangle$ case and return to our previous method of solving for a general form of m, finding the fundamental unit, then proving the existence of a prime f that gives us $H_d(f) = 1$. Because of the two degrees of freedom, we are unable to complete all these steps for the general case and will therefore look at only a few special cases.

Theorem 8.1. *The continued fraction* $\langle n, \overline{a, b, a, 2n}\rangle$ *is equal to* $\sqrt{n^2 + \dfrac{2abn + 2n + b}{a^2b + 2a}}$ *where* $a^2b+2a \mid 2abn+2n+b$ *and* $n^2 + \dfrac{2abn + 2n + b}{a^2b + 2a}$ *is not a perfect square.*

The proof of this theorem is of the same form as the corresponding proofs in the previous cases but is omitted here due to the complexity of the resulting quadratic equation.

We continue with this case by finding the fundamental unit.

Theorem 8.2. *Let* $\sqrt{m} = \langle n, \overline{a, b, a, 2n}\rangle = \sqrt{n^2 + \dfrac{2abn + 2n + b}{a^2b + 2a}}$. *Then the fundamental unit* $\varepsilon_m = (a^2bn + a^2 + b^2 + an + 1) + (a^2b + 2a)\sqrt{m}$.

Proof. □

We simplify this into a few special subcases, looking at only particular values of a and b.

In each case, we have simplified the problem to proving that there exists a prime f that divides m and does not divide the denominator of the fraction of the general form of m. It is understood that a relative class number of 1 will directly follow from proving the existence of such an f, using the same logic as in Theorems 7.3 and 7.8.

8.1. **The** $\langle n, \overline{a, b, a, 2n}\rangle$ **case when** $b = 1$. We will begin with the subcase when $b = 1$ and start by using Theorem **??** to find the fundamental unit.

Corollary 8.3 (of Theorem **??**)**.** *If* $\sqrt{m} = \langle n, \overline{a, 1, a, 2n}\rangle = \sqrt{n^2 + \dfrac{2an + 2n + 1}{a^2 + 2a}}$, *then the fundamental unit* $\varepsilon_m = (a^2n + a^2 + an + 2) + (a^2 + 2a)\sqrt{m}$.

We will now skip to proving the existence of an f that will give us $H_d(f) = 1$.

Theorem 8.4. *If* $\sqrt{m} = \langle n, \overline{a, 1, a, 2n}\rangle = \sqrt{n^2 + \dfrac{2an + 2n + 1}{a^2 + 2a}}$, *then a is odd and there exists a prime f such that $f \mid m$ and $f \nmid a^2 + 2a$.*

Proof. Let $k = \dfrac{2an + 2n + 1}{a^2 + 2a}$. If a were even, then the denominator would have to be even. But the numerator is always odd, so this would make k not an integer. Thus, a must be odd.

Now, since $k \in \mathbb{Z}$,

$$\begin{aligned} 2an + 2n + 1 &\equiv 0 \pmod{a^2 + 2a} \\ 2n(a+1) &\equiv -1 \pmod{a^2 + 2a} \\ &\equiv -a^2 - 2a - 1 \pmod{a^2 + 2a} \\ &\equiv -(a+1)^2 \pmod{a^2 + 2a} \\ n &\equiv -\left(\frac{a+1}{2}\right) \pmod{a^2 + 2a} \end{aligned}$$

So $n = l(a^2 + 2a) - \frac{a+1}{2}$ for some positive integer l and $m = n^2 + k = \left(l(a^2 + 2a) - \frac{a+1}{2}\right)^2 + k > a^2 + 2a$. Thus, there must be a prime f that divides m and does not divide $a^2 + 2a$. □

8.2. **The $\langle n, \overline{a, b, a, 2n}\rangle$ case when $b = 2$.** The next subcase we will look at is when $b = 2$ and we will again begin by finding the fundamental unit from our earlier theorem.

Corollary 8.5 (of Theorem **??**). *If $\sqrt{m} = \langle n, \overline{a, 2, a, 2n}\rangle = \sqrt{n^2 + \frac{4an + 2n + 2}{2a^2 + 2a}}$, then the fundamental unit $\varepsilon_m = (2a^2 n + a^2 + an + 3) + (2a^2 + 2a)\sqrt{m}$.*

We will now prove the existence of an f that will give us a relative class number 1, just as we did in the previous subcase.

Theorem 8.6. *If $\sqrt{m} = \langle n, \overline{a, 2, a, 2n}\rangle = \sqrt{n^2 + \frac{4an + 2n + 2}{2a^2 + 2a}}$, then there exists a prime f such that $f \mid m$ and $f \nmid 2a^2 + 2a$.*

Proof. Let $k = \frac{4an + 2n + 2}{2a^2 + 2a}$. Since $k \in \mathbb{Z}$,

$$\begin{aligned} 4an + 2n + 2 &\equiv 0 \pmod{2a^2 + 2a} \\ 2n(2a+1) &\equiv -2 \pmod{2a^2 + 2a} \\ n(2a+1) &\equiv -1 \pmod{a^2 + a} \\ n(2a+1)(2a+1) &\equiv -(2a+1) \pmod{a^2 + a} \\ n(4a^2 + 4a + 1) &\equiv -2a - 1 \pmod{a^2 + a} \\ n &\equiv -2a - 1 + a^2 + a \pmod{a^2 + a} \\ &\equiv a^2 - a - 1 \pmod{a^2 + a} \end{aligned}$$

So $n = l(a^2 + a) + a^2 - a - 1$ for some integer $l \geq 0$ and $m = n^2 + k = (l(a^2 + a) + a^2 - a - 1)^2 + k > 2a^2 + 2a$. Thus, there must be a prime f that divides m and does not divide $2a^2 + 2a$. □

8.3. **The $\langle n, \overline{a, b, a, 2n}\rangle$ case when $a = 1$.** The next subcase is when $a = 1$ and we first find the fundamental unit.

Corollary 8.7 (of Theorem **??**)**.** *If $\sqrt{m} = \langle n, \overline{1, b, 1, 2n}\rangle = \sqrt{n^2 + \dfrac{2bn + 2n + b}{b + 2}}$, then the fundamental unit $\varepsilon_m = (bn + n + b^2 + 2) + (b + 2)\sqrt{m}$.*

We now prove the existence of an f that will give us a relative class number of 1, just as we did in the previous subcases.

Theorem 8.8. *If $\sqrt{m} = \langle n, \overline{1, b, 1, 2n}\rangle = \sqrt{n^2 + \dfrac{2bn + 2n + b}{b + 2}}$, then there exists a prime f such that $f \mid m$ and $f \nmid b + 2$.*

Proof. Let $k = \dfrac{2bn + 2n + b}{b + 2}$. Since $k \in \mathbb{Z}$,

$$\begin{aligned} 2bn + 2n + b &\equiv 0 \pmod{b + 2} \\ 2n(b + 1) &\equiv 2 \pmod{b + 2} \\ -2n &\equiv 2 \pmod{b + 2} \\ 2n &\equiv -2 \pmod{b + 2} \end{aligned}$$

If b is odd we get

$$n \equiv -1 \pmod{b + 2}$$

So $n = l(b + 2) - 1$ for some positive integer l and $m = n^2 + k = (l(b + 2) - 1)^2 + k > b + 2$.

On the other hand, if b is even we get

$$n \equiv -1 \pmod{\frac{b}{2} + 1}$$

So $n = l\left(\frac{b}{2} + 1\right) - 1$ for some integer $l > 1$ and $m = n^2 + k = \left(l\left(\frac{b}{2} + 1\right) - 1\right)^2 + k > b + 2$. Thus, in either case there must be a prime f that divides m and does not divide $b + 2$. ☐

8.4. **The $\langle n, \overline{a, b, a, 2n}\rangle$ case when $a = 2$.** Our final subcase is when $a = 2$ and we once again find the fundamental unit from Theorem **??** first.

Corollary 8.9 (of Theorem **??**). *If* $\sqrt{m} = \langle n, \overline{2, b, 2, 2n} \rangle = \sqrt{n^2 + \dfrac{4bn + 2n + b}{4b + 4}}$, *then the fundamental unit* $\varepsilon_m = (4bn + 2n + b^2 + 5) + (4b + 4)\sqrt{m}$.

We now guarantee a relative class number of 1 in the same manner as the previous three subcases.

Theorem 8.10. *If* $\sqrt{m} = \langle n, \overline{2, b, 2, 2n} \rangle = \sqrt{n^2 + \dfrac{4bn + 2n + b}{4b + 4}}$, *then* b *is even and there exists a prime* f *such that* $f \mid m$ *and* $f \nmid 4b + 4$.

Proof. Let $k = \dfrac{4bn + 2n + b}{4b + 4}$. The denominator is always even and $k \in \mathbb{Z}$, so we know the numerator must be even as well. Thus b must be even.

Now since $k \in \mathbb{Z}$,

$$\begin{aligned}
4bn + 2n + b &\equiv 0 \pmod{4b + 4} \\
2n(2b + 1) &\equiv -b \pmod{4b + 4} \\
n(2b + 1) &\equiv -\frac{b}{2} \pmod{2b + 2} \\
-n &\equiv -\frac{b}{2} \pmod{2b + 2} \\
n &\equiv \frac{b}{2} \pmod{2b + 2}
\end{aligned}$$

So $n = l(2b+2) + \dfrac{b}{2}$ for some integer $l > 0$ and $m = n^2 + k = \left(l(2b + 2) - \dfrac{b}{2}\right)^2 + k > 4b + 4$. Thus, there must be a prime f that divides m and does not divide $4b + 4$. ☐

We have now shown that for any m such that $\sqrt{m} = \langle n, \overline{a, b, a, 2n} \rangle$ for either $b = 1$, $b = 2$, $a = 1$, or $a = 2$, there must exists an f such that $H_d(f) = 1$.

We now present one final case and proceed to first solve its continued fraction for a general form of m.

Theorem 8.11. *The continued fraction* $\langle n, \overline{a, b, b, a, 2n} \rangle$ *is equal to* $\sqrt{n^2 + \dfrac{2ab^2n + 2an + 2bn + b^2 + 1}{a^2b^2 + 2ab + a^2 + 1}}$ *where* $a^2b^2 + 2ab + a^2 + 1 \mid 2ab^2n + 2an + 2bn + b^2 + 1$ *and* $n^2 + \dfrac{2ab^2n + 2an + 2bn + b^2 + 1}{a^2b^2 + 2ab + a^2 + 1}$ *is not a perfect square.*

Once again, the proof is omitted due to the difficulty of the quadratic equation involved and this case is too difficult to examine all at once, so we will only look at the subcase when $b = 1$.

Additionally, we will immediately prove the existence of an f that divides m and does not divide the denominator of the corresponding fraction in the general form of m, with the assumption that this denominator is the y-term of the fundamental unit and will therefore give us a relative class number of 1 by the logic provided in previous cases.

Theorem 8.12. *If* $\sqrt{m} = \langle n, \overline{a, 1, 1, a, 2n} \rangle = \sqrt{n^2 + \frac{4an + 2n + 2}{2a^2 + 2a + 1}}$, *then there exists a prime* f *such that* $f \mid m$ *and* $f \nmid 2a^2 + 2a + 1$.

Proof. Let $m - n^2 = \frac{4an + 2n + 2}{2a^2 + 2a + 1} = k$. Since $k \in \mathbb{Z}$,

$$\begin{aligned}
4an + 2n + 2 &\equiv 0 \pmod{2a^2 + 2a + 1} \\
n(4a + 2) &\equiv -2 \pmod{2a^2 + 2a + 1} \\
&\equiv 4(2a^2 + 2a + 1) - 2 \pmod{2a^2 + 2a + 1} \\
&\equiv 8a^2 + 8a + 2 \pmod{2a^2 + 2a + 1} \\
&\equiv (2a + 1)(4a + 2) \pmod{2a^2 + 2a + 1} \\
n &\equiv 2a + 1 \pmod{2a^2 + 2a + 1}
\end{aligned}$$

So $n = 2a + 1 + l(2a^2 + 2a + 1)$ for some nonnegative integer l and $m = n^2 + k = (2a + 1 + l(2a^2 + 2a + 1))^2 + k > 2a^2 + 2a + 1$. Thus, there must be a prime that divides m and does not divide $2a^2 + 2a + 1$. $\square$

We have now proven that for any m value such that $\sqrt{m} = \langle n, \overline{a, 1, 1, a, 2n} \rangle$, there exists an f such that $H_d(f) = 1$.

9. Conclusion and Future Work

We have now shown that a relative class number of 1 exists for continued fraction expansions of the form $\langle n, \overline{2n} \rangle$, $\langle n, \overline{a, 2n} \rangle$, $\langle n, \overline{a, a, 2n} \rangle$, $\langle n, \overline{a, a, \ldots, a, 2n} \rangle$, $\langle n, \overline{a, 1, a, 2n} \rangle$, $\langle n, \overline{a, 2, a, 2n} \rangle$, $\langle n, \overline{1, b, 1, 2n} \rangle$, $\langle n, \overline{2, b, 2, 2n} \rangle$, and $\langle n, \overline{a, 1, 1, a, 2n} \rangle$. To give some perspective, these cover 39 out of the 60 square-free m values less than 100.

In addition, for the general $\langle n, \overline{a, b, a, 2n} \rangle$ and $\langle n, \overline{a, b, b, a, 2n} \rangle$ cases that we could not solve directly, we wrote a program in Python to check for the possibility that m divides the denominator of the corresponding fraction in the general form of m. If this were to happen, we would not be able to find an f that divides m and does not divide the denominator, so we would be unable to ensure a relative class number of 1. For both cases, we checked m values up to 2 million for this property and did not find one.

Future research in this area might include continuing our proof technique for larger continued fraction expansions, finding a way to generalize the $\langle n, \overline{a, b, a, 2n} \rangle$ and $\langle n, \overline{a, b, b, a, 2n} \rangle$ cases based on the simple subcases we solved (i.e. by induction), or finding a new way to handle these larger cases.

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