Oscillations of a Water Balloon

Sven Isaacson

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Outline

My Goal

- To model the waves which form on the surface of a water balloon impinging on a surface
    - Look at acoustic (pressure) waves created within the water balloon
    - Look at waves formed from deformation of the balloon surface

Figure : Waves formed on a water balloon surface

Update

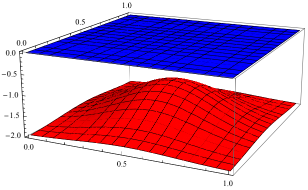

Figure : A travelling Gaussian isobar impinging from below a membrane

- Previous approach looked at an acoustic driving force driving oscillations on a membrane
- This is mathematically complicated: two coupled PDEs (the acoustic pressure wave, and the wave equation on the surface)
- Better approach: try modelling the surface force as the surface tension of a non-wetting droplet
- This is governed by the Young-Laplace Equation

Brief Review

Fluid mechanics: describe the velocity of "elements" of the fluid, $\vec{u}$

If irrotational flow: $\nabla \times \vec{u} = 0$, therefore $\vec{u} = \nabla\psi$

ψ is called the **velocity potential** and it satisfies Laplace's Equation

$$\nabla^2\psi = 0$$

Goal: Solve the Laplace equation for the a droplet.

- Velocity potential of fluid at surface of balloon will give velocity of balloon surface
- Need a boundary condition to solve the Laplace Equation

Young-Laplace Equation

The Young-Laplace Equation describes the pressure difference at the surface between two fluid media:

$$\Delta p = \gamma \Omega$$

- $\Delta p = p_1 - p_2$ where p_1 is pressure in medium 1 and p_2 is pressure in medium 2
- γ is the surface tension (units J/m^2 or N/m)
- Ω is the the curvature $(1/R_1 + 1/R_2)$ where R_1 and R_2 are the radii of curvature of the surface in two orthogonal directions

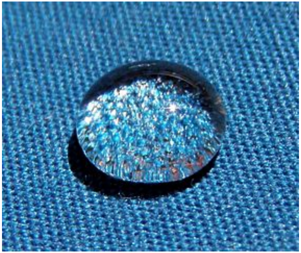

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Figure : A fluid-fluid interface between water and air ($\gamma \approx 72$mN/m)

A Slightly Deformed Sphere

Need to calculate the curvature of a sphere that is slightly deformed

Consider radius of slightly deformed sphere to be

$$r(\theta, \phi) = R + \zeta(\theta, \phi)$$

- R is the original radius
- ζ is a small deviation from R

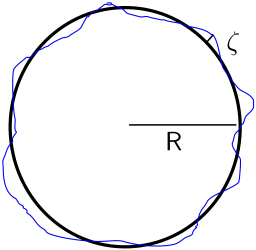

Figure : Near-sphere, with slight changes in radius ζ

What is $\frac{1}{R_1} + \frac{1}{R_2}$?

Can be calculated by equating the infinitesimal change in the surface area

$$\delta A = \iint \delta\zeta \left(\frac{1}{R_1} + \frac{1}{R_2} \right) dA$$

$\delta\zeta$ – small change in radius.
Alternatively, calculating the surface area of the deformed sphere:

$$A = \iint (R + \zeta)\sqrt{1 + \nabla^2 r}\delta\zeta dA$$

which for small change $\delta\zeta$ becomes

$$\delta A = \iint \left[\frac{2}{R} - \frac{2\zeta}{R^2} - \frac{1}{R^2} \left(\frac{1}{\sin^2\theta} \frac{\partial^2\zeta}{\partial\phi^2} + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial\zeta}{\partial\theta} \right) \right) \right] \delta\zeta dA$$

equating the integrands we get...

Surface Pressure and Fluid Pressure

Young-Laplace Equation becomes

$$\Delta p = p_f - p_{air} = \gamma \left[\frac{2}{R} - \frac{2\zeta}{R^2} - \frac{1}{R^2} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial\zeta}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2\zeta}{\partial\phi^2} \right) \right]$$

- p_{air} is constant, ambient
- $p_f = -\rho \frac{\partial\psi}{\partial t}$

At the surface $\partial\zeta/\partial t = \partial\psi/\partial r$. Differentiate the above w.r.t. time and substitute:

The boundary condition

$$\rho \frac{\partial^2\psi}{\partial t^2} - \frac{\gamma}{R^2} \left[2\frac{\partial\psi}{\partial r} + \frac{\partial}{\partial r} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial\psi}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2\psi}{\partial\phi^2} \right) \right] = 0$$

Contact Pressure

The pressure on the surface isn't p_{air} at every point of the sphere. At the bottom there is a Dirac delta pressure

$$P_f = \delta(r = R, \theta = \pi, \phi = 0)$$

this changes the boundary condition equation (adds an extra term)

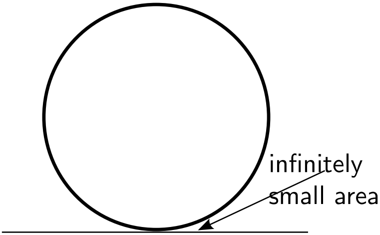

Figure : A sphere droplet resting on a plane

Solution of Laplace's Equation

Look for a solution

$$\psi = \exp(-i\omega t) f(r, \theta, \phi)$$

so

$$\begin{aligned} \nabla^2 \psi &= 0 \\ \nabla^2 (\exp(-i\omega t) f(r, \theta, \phi)) &= \\ \exp(-i\omega t) \nabla^2 f(r, \theta, \phi) &= \\ \nabla^2 f(r, \theta, \phi) &= 0 \end{aligned}$$

so f must solve Laplace's Equation.

Spherical Harmonics

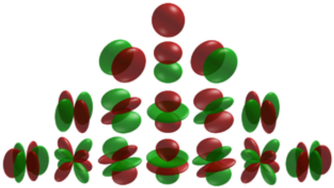

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Figure : The first 4 sets of spherical harmonics

Well known solution to Laplace's Equation in spherical coordinates:

$$f(r, \theta, \phi) = r^l Y_{l,m}(\theta, \phi)$$

Also, $Y_{l,m}$ are eigenfunctions of the Laplacian:

$$\nabla^2 Y_{l,m} = -l(l+1) Y_{l,m}$$

Plugging in our solution

The boundary condition

$$\rho \frac{\partial^2 \psi}{\partial t^2} - \frac{\gamma}{R^2} \left[2\frac{\partial \psi}{\partial r} + \frac{\partial}{\partial r} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 \psi}{\partial \phi^2} \right) \right] = 0$$

with

$$\psi = \exp(-i\omega t) r^l Y_{l,m}(\theta, \phi)$$

reduces to

$$\omega_l^2 = \frac{\gamma l(l-1)(l+2)}{\rho R^3}$$

or, when the expansion of the contact force is included

$$\omega_l^2 = \frac{\gamma}{\rho R^3} \frac{l(l-1)(l+2)}{1 + \sqrt{(2l+1)/4\pi}}$$

Summary

- Surface effects should be treated as surface tensions, to avoid two coupled PDEs
- Young-Laplace equation governs pressure differences caused by surface tension
- The Y-L equation can be used to get a boundary condition of the Laplace equation for fluid velocity potential

Conclusions

There are some problems with this model

- Applied pressure is not just at a point, but grows with time
- Difficult to determine "surface tension" of a balloon – wouldn't expect this to be equal to the elastic tension
- This is theory is for *small* droplets for which gravity is negligible to capillary action

However, this my best attempt yet

- Neatly ties together the surface term and the internal velocity field
- Reduces to the easily solved Laplace equation, for the velocity potential

Future Work

- Account for gravity waves in the water balloon
- Treat contact force as an expanding area as a function of time, rather than point
- Compare measured values to predicted

Acknowledgements and References

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