

Describing a Combinatorics Problem with a System of Polynomial Equations

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ABSTRACT

This paper provides a method to describe and solve a combinatorics problem using systems of polynomial equations. These systems, however, are too large to be solved by hand. The goal of this paper is to give the reader two techniques to solve these systems. The first technique uses Buchberger’s Algorithm to find a Gröbner basis for the system. The second technique addresses and solves the problem if finding a Gröbner basis is computationally difficult.



1. INTRODUCTION

In 1970, Milton Bradley(c) created a game played on a hexagon-shaped grid called ‘Drive Ya Nuts.’ The game consists of seven hexagonal nuts, each having a unique arrangement of the numbers one through six on each side. The object of the game is to arrange the nuts on the grid in such a way that adjacent sides of the nuts have matching numbers. Up to rotation of the entire game board, there are possible 235,146,240 ways to place the nuts. Suppose that we were able to go through every combination and check if it were a solution every second, then it would take 7.46 years to find all solutions. In this paper we will determine how many of these combinations are solutions. Disregard-

ing a brute force attempt to find all solutions, we begin by describing this game by a system of polynomial equations.

2. DESCRIBING THE GAME

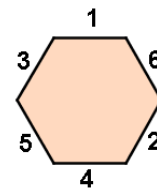


Figure 1: Nut B_0 in initial rotation state

2.1 Notation and Description

Each nut has a particular ordering of one through six. I will refer to the ordering of a specific nut by a 6-tuple, headed by any number with subsequent numbers listed clockwise. The first entry of the 6-tuple will correspond to number located on the north side of the nut, which we call position 0. The second entry will correspond to the number located on the east-north-east side of the nut, which we call position 1. Following entries will correspond to the next side moving clockwise up to position 5 corresponding to west-north-west.

If the zero entry of the 6-tuple is 1, then we shall call that the *initial rotation state of the nut*. For instance, in Figure 1, the second entry of B_0 is 2 and the fourth entry is 5. In a randomly assigned order, here are the definitions for each nut in the initial rotation state: $B_0 = (1, 6, 2, 4, 5, 3)$, $B_1 = (1, 4, 6, 2, 3, 5)$, $B_2 = (1, 6, 5, 3, 2, 4)$, $B_3 = (1, 4, 3, 6, 5, 2)$, $B_4 = (1, 2, 3, 4, 5, 6)$, $B_5 = (1, 6, 4, 2, 5, 3)$, $B_6 = (1, 6, 5, 4, 3, 2)$.

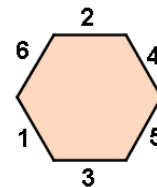


Figure 2: Nut B_0 rotated 4 times

Each nut also has six distinct rotational states. A rotation of one rotates the nut sixty degrees clockwise. A rotation of two rotates the nut 120 degrees clockwise. Up to $n = 5$ rotations, a rotation of n rotates the nut $60n$ degrees. For

example, B_0 rotated four times would look like Figure 2. The corresponding 6-tuple for B_0 rotated four times would be (2, 4, 5, 3, 1, 6).

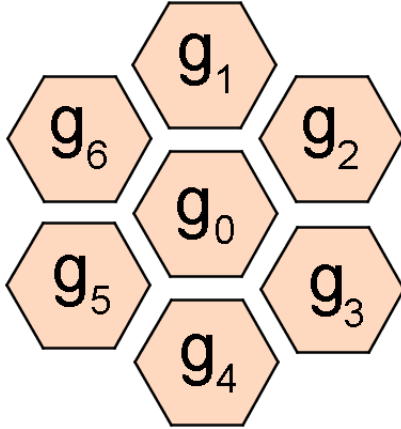


Figure 3: Location of g_i on the grid

To describe the location of a nut on the game board, each location g_i is given a value. Position of the nuts placed on the grid is denoted by a number 0 through 6. Nut position 0 is in the center of the grid. Nut position 1 is north on the grid. Nut position 2 is east north-east on the grid and similarly up to nut position 6 located north west-north on the grid.

We say that $p_i = m$ signifies that B_i is rotated 60 degrees m times and $g_j = n$ signifies that B_n is placed on the grid at position j . An instance of this is shown by if $g_4 = 3$ and $p_3 = 1$, then B_3 rotated once would be placed at position 4, corresponding to 3 in Figure 3. These variables will be used again in §3.

3. DESCRIBING THE NUTS

3.1 The Rotation Equations

Our goal is to define a function $F(g, p, x)$ that describes the possible states of the nuts in the game. In particular, $F(g, p, x)$ gives the value at any B_g under any rotation p at position x . In the following sections, we will be creating smaller functions and combining them to create the final $F(g, p, x)$ function.

We create functions $f_{i,j}(x)$, where i denotes nut B_i and j denotes the j^{th} rotation of B_i . We start by describing $f_{0,0}(x)$ that describes B_0 under the initial rotation; in other words, if we plug in a position $x \in \{0, 1, 2, 3, 4, 5\}$, $f_{0,0}(x)$ will satisfy the following conditions: $f_{0,0}(0) = 1$, $f_{0,0}(1) = 6$, $f_{0,0}(2) = 2$, $f_{0,0}(3) = 4$, $f_{0,0}(4) = 5$, and $f_{0,0}(5) = 3$.

Consider $f_{0,0}(x)$ defined by

$$f_{0,0}(x) = \sum_{i=0}^5 c_i x(x-1) \dots \widehat{(x-i)} \dots (x-5)$$

Since each term of $f_{0,0}(x)$ is zero except $c_i x \dots \widehat{(x-i)} \dots (x-5)$, the value of $f_{0,0}(x)$ can be controlled by the value of c_i . From our initial conditions, our final function is $f_{0,0}(x) = -\frac{1}{120}(x-1)(x-2)(x-3)(x-4)(x-5) + \frac{1}{4}x(x-2)(x-3)(x-$

$4)(x-5) - \frac{1}{6}x(x-1)(x-3)(x-4)(x-5) + \frac{1}{3}x(x-1)(x-2)(x-4)(x-5) - \frac{5}{24}x(x-1)(x-2)(x-3)(x-5) + \frac{3}{120}x(x-1)(x-2)(x-3)(x-4)$, or expanded becomes $f_{0,0}(x) = 1 + \frac{127x}{5} - \frac{100x^2}{3} + \frac{127x^3}{8} - \frac{19x^4}{6} + \frac{9x^5}{40}$. Notice how Figure 4 satisfies our conditions.

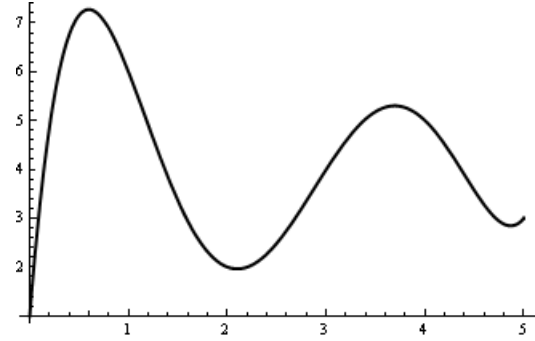


Figure 4: $f_{0,0}(x)$ on $[0, 5]$

Now we must repeat the process for $f_{0,1}(x)$ using the next clockwise rotation of nut B_0 , which corresponds to (3, 1, 6, 2, 4, 5). Since finding the c_i values can be long and tedious work, we streamline the method using matrices to find our expanded polynomials. Consider the function $f_{0,1}(x) = a_0x^5 + a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5$. We want to find values of $a_0, a_1, a_2, a_3, a_4, a_5$ such that we obtain our desired function. However, this is just a system of 6 equations and 6 unknowns. $f_{0,1}(0)$ gives us $a_5 = 3$, $f_{0,1}(1)$ gives us $a_0 + a_1 + a_2 + a_3 + a_4 + a_5 = 1$, $f_{0,1}(2)$ gives us $32a_0 + 16a_1 + 8a_2 + 4a_3 + 2a_4 + a_5 = 6$, and so on. Setting up the corresponding matrix equation, we get:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 32 & 16 & 8 & 4 & 2 & 1 \\ 243 & 81 & 27 & 9 & 3 & 1 \\ 1024 & 256 & 64 & 16 & 4 & 1 \\ 3125 & 625 & 125 & 25 & 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 6 \\ 2 \\ 4 \\ 5 \end{bmatrix}$$

By multiplying on the left by the inverse of the coefficient matrix, we arrive at the solution:

$$\begin{bmatrix} -\frac{1}{120} & \frac{1}{24} & -\frac{1}{12} & \frac{1}{12} & -\frac{1}{24} & \frac{1}{120} \\ \frac{1}{8} & -\frac{7}{12} & \frac{13}{12} & -1 & \frac{11}{24} & -\frac{1}{12} \\ -\frac{17}{24} & \frac{71}{24} & -\frac{59}{12} & \frac{49}{12} & -\frac{41}{24} & \frac{7}{24} \\ \frac{15}{8} & -\frac{77}{12} & \frac{107}{12} & -\frac{13}{2} & \frac{61}{24} & -\frac{5}{12} \\ -\frac{137}{60} & 5 & -5 & \frac{10}{3} & -\frac{5}{4} & \frac{1}{5} \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 6 \\ 2 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -\frac{53}{120} \\ \frac{137}{24} \\ -\frac{207}{8} \\ \frac{1147}{24} \\ -\frac{1751}{60} \\ 3 \end{bmatrix}$$

Therefore $f_{0,1}(x) = -\frac{53}{120}x^5 + \frac{137}{24}x^4 - \frac{207}{8}x^3 + \frac{1147}{24}x^2 - \frac{1751}{60}x + 3$. Repeat this process to find all $f_{i,j}(x)$ for $i \in \{0, 1, 2, 3, 4, 5, 6\}$, $j \in \{0, 1, 2, 3, 4, 5\}$.

3.2 Combining the Rotation Equations

Next we create a function $h_k(p, x)$ that gives the corresponding value at the x^{th} position at the p^{th} rotation state of B_k .

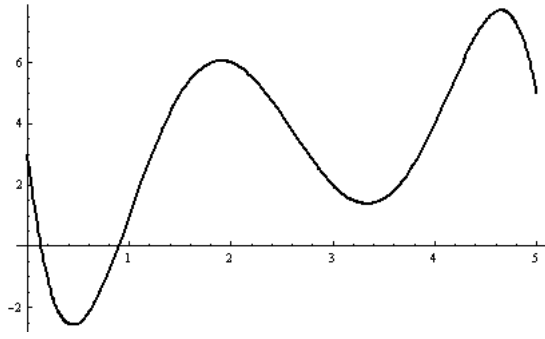


Figure 5: $f_{0,1}(x)$ on $[0, 5]$

For instance, $h_4(2, 5) = 4$ because $B_4 = (1, 2, 3, 4, 5, 6)$, rotated twice becomes $(5, 6, 1, 2, 3, 4)$, and the value at the fifth position is 4. Another example is $h_6(4, 0) = 5$, where $B_6 = (1, 6, 5, 4, 3, 2)$, rotated four times becomes $(5, 4, 3, 2, 1, 6)$, and the value at position zero is 5. Consider

$$h_k(p, x) = \sum_{i=0}^5 [d_i f_{k,i}(x)] p(p-1) \dots \widehat{(p-i)} \dots (p-5)$$

where d_i is a constant. All that is left is to find corresponding d_i values such that our function represents the entire nut.

If we let $p = 0$, then $h_0(0, x) = [d_0 f_{0,0}(x)](0-1)(0-2)(0-3)(0-4)(0-5)$. Because we do not want to change the value of $f_{0,0}(x)$ by a constant multiple, we must solve the equation $d_0(0-1)(0-2)(0-3)(0-4)(0-5) = 1$. In the case of $h_0(0, x)$, $d_0 = \frac{1}{-120}$. In general, we do not want to change the value of any $f_{j,i}$, so we set $d_i i(i-1) \dots \widehat{(i-i)} \dots (i-5) = 1$. Thus, $d_i = \frac{1}{i \dots (i-(i-1))(i-(i+1)) \dots (i-5)}$.

Expanded, our polynomial equation for h_0 is $h_0(p, x) = 1 - \frac{p}{4} + \frac{91p^2}{24} - \frac{7p^3}{4} + \frac{5p^4}{24} + \frac{127x}{5} - \frac{120237px}{400} + \frac{589913p^2x}{1440} - \frac{32467p^3x}{160} + \frac{61123p^4x}{1440} - \frac{7633p^5x}{2400} - \frac{100x^2}{3} + \frac{761957p^2x^2}{1440} - \frac{435481p^2x^2}{576} + \frac{220559p^3x^2}{576} - \frac{46643p^4x^2}{576} + \frac{17551p^5x^2}{2880} + \frac{127x^3}{8} - \frac{24401p^3x^3}{80} + \frac{129193p^2x^3}{288} - \frac{133019p^3x^3}{576} + \frac{14213p^4x^3}{2880} - \frac{10769p^5x^3}{2880} - \frac{19x^4}{6} + \frac{101947p^4x^4}{1440} - \frac{61223p^2x^4}{576} + \frac{31909p^3x^4}{576} - \frac{6877p^4x^4}{576} + \frac{2621p^5x^4}{2880} + \frac{9x^5}{40} - \frac{3427p^5x^5}{600} + \frac{6271p^2x^5}{720} - \frac{13199p^3x^5}{2880} + \frac{179p^4x^5}{180} - \frac{1097p^5x^5}{14400}$.

We again can verify our equation with several inputs, such as $h_0(1, 3) = 2$, $h_0(5, 2) = 4$, $h_0(2, 4) = 2$, and $h_0(4, 5) = 6$. In a similar way, we construct all $h_k(p, x)$.

3.3 The Final Equation

Our final task is to combine the h_0, h_1, \dots, h_6 polynomials into one massive polynomial $F(g, p, x)$ that gives us the value on nut B_g (defined in Section 2.1) at the p^{th} rotation state at the x^{th} position. Ideally, like the examples in §2.3, $F(4, 2, 5) = 4$ because the fifth position on B_4 rotated twice is 4 and $F(6, 4, 0) = 5$ because the zeroth position on B_6 rotated four times is 5. In a similar fashion to our previous methods, consider the polynomial

$$F(g, p, x) = \sum_{i=0}^6 [b_i h_i(p, x)] g(g-1) \dots \widehat{(g-i)} \dots (g-6)$$

where b_i is a constant. Again, we only need to choose the correct b_i value to get the polynomial we desire. Using meth-

ods similar to finding the d_i of the $h(p, x)$, we can easily find the values of the b_i . The final expanded $F(g, p, x)$ function is listed in the appendix; it has degree 16 and 245 monomials.

4. THE SOLUTION

4.1 The Adjacency System

Now that we have described all of the nuts by one large polynomial, we are able to describe how adjacency acts in this system of polynomials. We can encode the fact that the value of adjacent nuts are equal by setting some of these $F(g, p, x)$ polynomials with different g, p , and x values equal to each other.

First, in order to remove rotational redundancies from the game, we set the innermost nut, defined to be g_0 , to always have initial rotation ($p_0 = 0$) so we do not get similar solutions based on rotations of g_0 . Since g_1 is directly north of g_0 , we must set the rotation state of g_1 to $p_1 = 3$ so that the ones will be adjacent. Furthermore, since p_1 is always 3, we do not need to have p_1 in the final system of equations. The equation describing this is $F(g_0, 0, 0) - F(g_1, 3, 3) = 0$. Geometrically, this equation ensures that the north edge (position 0) of the nut at location g_0 with initial rotation and the south edge (position 3) of location g_1 with third rotation, and $F(g_0, 0, 0) - F(g_1, 3, 3) = 0$ ensures that the two edges have the same value. Of course this value will be 1.

Next, considering nut g_2 , the equations $F(g_0, 0, 1) - F(g_2, p_2, 4) = 0$ and $F(g_1, 3, 2) - F(g_2, p_2, 5) = 0$ are added into the system. Geometrically, g_0 with initial rotation at position 1 shares common value with g_2 with rotation p_2 at position 4. Similarly, g_1 with rotation 3 at position 2 shares common value with g_2 with rotation p_2 at position 5. Sequentially adding all of the nuts, we arrive at twelve distinct equations describing adjacency.

4.2 Domain and Uniqueness of Solution

We must take into consideration that the adjacency solutions must have integer values $g_i \in [0, 6]$ and $p_j \in [0, 5]$. Therefore, we must add in the following equations into our system to ensure integer solutions: $g_0(g_0-1)(g_0-2)(g_0-3)(g_0-4)(g_0-5)(g_0-6) = 0, \dots, g_6(g_6-1)(g_6-2)(g_6-3)(g_6-4)(g_6-5)(g_6-6) = 0, p_2(p_2-1)(p_2-2)(p_2-3)(p_2-4)(p_2-5) = 0, \dots, p_5(p_5-1)(p_5-2)(p_5-3)(p_5-4)(p_5-5) = 0$. The g_i equations will only zero out for $g_i \in \{0, 1, 2, 3, 4, 5, 6\}$ and the p_j equations will only zero out for $p_j \in \{0, 1, 2, 3, 4, 5\}$, hence forcing the solutions to these equations to be in our restricted integer domain.

Also, since each nut can be used only once, we must ensure that each g_i is unique. The following equation will handle that: $(g_0 - g_1)(g_0 - g_2)(g_0 - g_3)(g_0 - g_4)(g_0 - g_5)(g_0 - g_6)(g_1 - g_2)(g_1 - g_3)(g_1 - g_4)(g_1 - g_5)(g_1 - g_6)(g_2 - g_3)(g_2 - g_4)(g_2 - g_5)(g_2 - g_6)(g_3 - g_4)(g_3 - g_5)(g_3 - g_6)(g_4 - g_5)(g_4 - g_6)(g_5 - g_6)a - 1 = 0$, where a is another variable. Notice that if $g_i - g_j = 0$ for some $i \neq j$, then $-1 = 0$. Thus a ensures that the product of all the $g_i - g_j$ is 1. For efficiency, we break up the uniqueness equation into several, smaller similar equations: $(g_0 - g_1)(g_0 - g_2)(g_0 - g_3)(g_0 - g_4)(g_0 - g_5)(g_0 - g_6)a - 1 = 0$, $(g_1 - g_2)(g_1 - g_3)(g_1 - g_4)(g_1 - g_5)(g_1 - g_6)b - 1 = 0$, $(g_2 - g_3)(g_2 - g_4)(g_2 - g_5)(g_2 - g_6)(g_3 - g_4)(g_3 - g_5)(g_3 - g_6)c - 1 = 0$, and $(g_4 - g_5)(g_4 - g_6)(g_5 - g_6)d - 1 = 0$, where a, b, c , and d are variables.

4.3 Finding a Solution

We have 27 equations of degree 16 or less in 16 variables with rational coefficients and we want to find a solution to system. If we multiply an equation in the system by some $K \in \mathbb{R}[g_0, g_1, g_2, g_3, g_4, g_5, g_6, p_2, p_3, p_4, p_5, a, b, c, d]$ or add any two of the equations in the system together, the solution will remain unchanged. Thus it is natural to consider the ideal generated by the adjacency equations, domain equations, and uniqueness equations. We will call the ideal generated by our system $\langle I \rangle$.

The variety of $\langle I \rangle$, $V(\langle I \rangle)$, corresponds to $\{(g_0, g_1, g_2, g_3, g_4, g_5, g_6, p_2, p_3, p_4, p_5, a, b, c, d) \in \mathbb{R}^{16} : r_i(g_0, g_1, g_2, g_3, g_4, g_5, g_6, p_2, p_3, p_4, p_5, a, b, c, d) = 0 \forall r_i \in I\}$. In other words, this is solution set of our system of equations. But $\langle I \rangle$ is too complicated to analyze and find all the zeros, so we would like to find another generating set of equations $G \subset \mathbb{R}[g_0, g_1, g_2, g_3, g_4, g_5, g_6, p_2, p_3, p_4, p_5, a, b, c, d]$ that generates $\langle I \rangle$. Proposition 4 on page 32 of [2] guarantees that if $\langle I \rangle = \langle G \rangle$, then $V(\langle I \rangle) = V(\langle G \rangle)$. The converse of the statement is not true because $V(\langle x \rangle) = V(\langle x^2 \rangle)$ which is just $x = 0$, but $\langle x \rangle \neq \langle x^2 \rangle$.

A Gröbner basis is ideal for this situation because Gröbner bases are a nice generating set of an ideal. Most of the time we can read off or easily determine the variety from the Gröbner basis. For example, find all points in $V(K)$ where $K = \langle x^2 + y^2 + z^2 - 1, x^2 + z^2 - y, x - z \rangle \subset \mathbb{C}[x, y, z]$. The solution is not obvious, but if we take the Gröbner basis of K with respect to lexicographic monomial order, we get $\langle x - z, -y + 2z^2, z^4 + \frac{1}{2}z^2 - \frac{1}{4} \rangle$. One of the polynomials of the Gröbner basis is solely in z , and solving for z we get $z = \pm \frac{1}{2} \sqrt{\pm \sqrt{5} - 1}$. This gives us four values for z , and by substitution, we can obtain the rest of the points in $V(K)$. For more background on Gröbner bases, consult the text [2].

There exists an algorithm for converting a basis into a Gröbner basis for an ideal. By means of a computer algebra system, we attempted to calculate the Gröbner basis, G , of I . Due to the lack of high-powered computing, we were unable to do so. We can, however, compute $V(\langle I \rangle)$ and we will do so in section 4.4. Using $V(\langle I \rangle)$, we can easily find a Gröbner basis for the ideal generated by $V(\langle I \rangle)$. Finally, we will give the relationship of this ideal to our original one.

4.4 Calculating $V(\langle I \rangle)$

The following theorem gives all solutions to ‘Drive Ya Nuts.’

THEOREM 4.1. *The solutions to ‘Drive Ya Nuts’ are $V(\langle I \rangle) = \{g_0 = 0, g_1 = 1, g_2 = 2, p_2 = 3, g_3 = 3, p_3 = 0, g_4 = 4, p_4 = 3, g_5 = 5, p_5 = 3, g_6 = 6, p_6 = 4, a = \frac{1}{720}, b = \frac{-1}{120}, c = \frac{-1}{24}, d = \frac{-1}{12}\}$.*

PROOF. We will be using Mathematica’s **Reduce** function to ‘build-up’ a solution from partial solutions. The way that **Reduce** works is that **Reduce[expr, vars]** reduces the statement **expr** by solving equations or inequalities for **vars** and eliminating quantifiers. Starting with **Reduce[(g_0(g_0-1)(g_0-2)(g_0-3)(g_0-4)(g_0-5)(g_0-6) = 0, g_1(g_1-1)(g_1-2)(g_1-3)(g_1-4)(g_1-5)(g_1-6) = 0, g_2(g_2-1)(g_2-2)(g_2-3)(g_2-4)(g_2-5)(g_2-6) = 0, p_2(p_2-1)(p_2-2)(p_2-3)(p_2-4)(p_2-5) = 0, g_0 ≠ g_1, g_0**

≠ g_2, g_1 ≠ g_2, F(g_0,0,0)-F(g_1,3,3) = 0, F(g_0,0,1)-F(g_2,p_2,4) = 0, F(g_1,3,2) - F(g_2,p_2,5) = 0}, {g_0,g_1,g_2,p_2}, **Reduce** will give us a partial solution set of the first three nuts, which is $\{(g_0==0 \ \&\& \ g_1==1 \ \&\& \ g_2==2 \ \&\& \ p_2==3) \ || \ (g_0==0 \ \&\& \ g_1==1 \ \&\& \ g_2==3 \ \&\& \ p_2==1) \ || \ (g_0==0 \ \&\& \ g_1==2 \ \&\& \ g_2==5 \ \&\& \ p_2==3) \ || \ (g_0==0 \ \&\& \ g_1==3 \ \&\& \ g_2==1 \ \&\& \ p_2==2) \ || \ (g_0==0 \ \&\& \ g_1==6 \ \&\& \ g_2==1 \ \&\& \ p_2==2) \ || \ (g_0==1 \ \&\& \ g_1==0 \ \&\& \ g_2==3 \ \&\& \ p_2==3) \ || \ (g_0==1 \ \&\& \ g_1==0 \ \&\& \ g_2==6 \ \&\& \ p_2==1) \ || \ (g_0==1 \ \&\& \ g_1==3 \ \&\& \ g_2==5 \ \&\& \ p_2==2) \ || \ \dots \ || \ (g_0==6 \ \&\& \ g_1==3 \ \&\& \ g_2==1 \ \&\& \ p_2==2)\}$.

We will proceed to use **Reduce** again including the partial solution set, g_3 and p_3 domain restrictions, $F(g_3, p_3, 5) - F(g_0, 0, 2)$, $F(g_3, p_3, 0) - F(g_2, p_2, 3)$, and the nut inequalities. By repeating this process, **Reduce** will narrow the possible solutions until all combinations have been exhausted, leaving only a set of full solutions. The final set of solutions is $g_0==0 \ \&\& \ g_1==1 \ \&\& \ g_2==2 \ \&\& \ p_2==3 \ \&\& \ g_3==3 \ \&\& \ p_3==0 \ \&\& \ g_4==4 \ \&\& \ p_4==3 \ \&\& \ g_5==5 \ \&\& \ p_5==3 \ \&\& \ g_6==6 \ \&\& \ p_6==4$. However, this method models a ‘brute force’ attempt to find all solutions. **Reduce** starts with the three nuts and calculates a set of partial solutions. Then the next iteration of **Reduce** takes another nut and calculates from next set of partial solutions using the partial solutions from the previous iteration. **Reduce** basically narrows down the initial 235, 146, 240 ways to place the nuts to 40 partial solutions, then to 37 partial solutions, then to 23, then 4, and finally one entire solution to the puzzle.



Figure 6: The only solution to Drive Ya Nuts

□

Now we define the ideal of the functions that vanish on our variety. Clearly, $I(V(\langle I \rangle)) = \langle g_0, g_1 - 1, g_2 - 2, g_3 - 3, g_4 - 4, g_5 - 5, g_6 - 6, p_2 - 3, p_3, p_4 - 3, p_5 - 3, p_6 - 4, a - \frac{1}{720}, b + \frac{1}{120}, c + \frac{1}{24}, d + \frac{1}{12} \rangle$, contains I . The basis here is a Gröbner basis.

THEOREM 4.2. *$\{g_0, g_1 - 1, g_2 - 2, g_3 - 3, g_4 - 4, g_5 - 5, g_6 - 6, p_2 - 3, p_3, p_4 - 3, p_5 - 3, p_6 - 4, a - \frac{1}{720}, b + \frac{1}{120}, c + \frac{1}{24}, d + \frac{1}{12}\}$ is a Gröbner basis for $I(V(I))$.*

PROOF. Using a computer algebra system, the Gröbner basis of $I(V(I))$ is $\langle 1+12d, 1+24c, 1+120b, -1+720a, -4+p_6, -3+p_5, -3+p_4, p_3, -3+p_2, -6+g_6, -5+g_5, -4+g_4, -3+g_3, -2+g_2, -1+g_1, g_0 \rangle$. \square

Our goal was to find a Gröbner basis for I , but at best we have found a Gröbner basis that contains I and has the same variety.

THEOREM 4.3. $\langle I \rangle \subset \langle 1+12d, 1+24c, 1+120b, -1+720a, -4+p_6, -3+p_5, -3+p_4, p_3, -3+p_2, -6+g_6, -5+g_5, -4+g_4, -3+g_3, -2+g_2, -1+g_1, g_0 \rangle$

PROOF. The proof follows directly from Lemma 7 on page 34 of [2]. \square

We hope in the future to be able to show that the above is an equality and show that $\langle 1+12d, 1+24c, 1+120b, -1+720a, -4+p_6, -3+p_5, -3+p_4, p_3, -3+p_2, -6+g_6, -5+g_5, -4+g_4, -3+g_3, -2+g_2, -1+g_1, g_0 \rangle$ is a Gröbner basis for our ideal.

5. SOLVING CIPRA'S PROBLEM

One interesting application to this technique of describing a combinatorics problem as a system of polynomial equations is Barry Cipra's Problem featured in [1]. There are sixteen distinct squares to be arranged on a four by four grid. Each square contains a distinct combination of a horizontal line through the center, a vertical line through the center, an upright diagonal through the center, and a down-right diagonal through the center. Each of these squares is to be placed on the grid, rotations not allowed, such that all horizontal, diagonal, and vertical lines are unbroken.

Solution redundancy is difficult to avoid in this puzzle because some of the squares are 90 degree and 180 degree rotations of other squares. Since some squares have four distinct rotations, other squares have two distinct rotations, and three squares do not have distinct rotations, then for simplicity, we will avoid the usage of rotations in this problem.

The polynomial for each square would be

$$h_j(x) = \sum_{i=0}^3 c_i x \dots (\widehat{x-i}) \dots (x-3)$$

where j is your self-defined square s_j and c_i controls the value of $h_j(x)$ at your defined position x : 1 (if a line is there) or 0 (if no line is there). For instance, in Figure 7, $h_5(0) = 1$, $h_5(1) = 1$, $h_5(2) = 0$, and $h_5(3) = 0$. Thus $h_5(x) = \frac{-1}{6}(x-1)(x-2)(x-3) + \frac{1}{2}x(x-2)(x-3) + 0x(x-1)(x-3) + 0x(x-1)(x-2) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + \frac{7}{6}x + 1$.

Adjacency works similarly in Cipra's Puzzle. Say that we have created a function $F(p, x)$, where $p_i = n$ is our pre-defined square s_n placed at location i on the grid and x is a boolean value of 1 (if a line is at position x) or 0 (if no line is at position x) for $x \in \{0, 1, 2, 3\}$. There are 42 adjacency equations in Cipra's Puzzle; here are the first

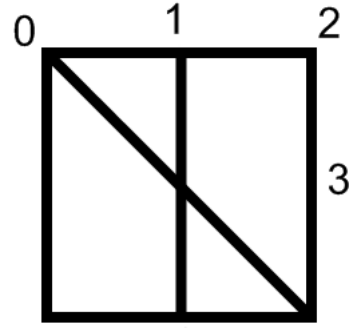


Figure 7: A square in Cipra's Puzzle

three equations for p_0, p_1, p_4, p_5 : $F(p_0, 3) - F(p_1, 3) = 0$, $F(p_0, 1) - F(p_4, 1) = 0$, and $F(p_0, 0) - F(p_5, 0) = 0$. The equation $F(p_0, 3) - F(p_1, 3) = 0$ corresponds to p_0 and p_1 sharing a horizontal line, $F(p_0, 1) - F(p_4, 1) = 0$ corresponds to p_0 and p_1 sharing a vertical line, and $F(p_0, 0) - F(p_5, 0) = 0$ corresponds to p_0 and p_1 sharing a diagonal down-right line.

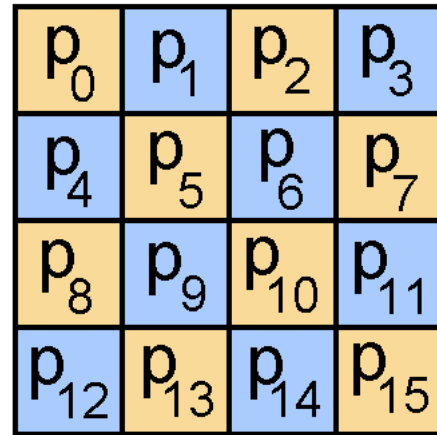


Figure 8: Locations of the p_i

We need to restrict the domain of the p_i to $\{0, 1, \dots, 15\}$, so we need equations like $p_0(p_0-1) \dots (p_0-14)(p_0-15)$, $p_1(p_1-1) \dots (p_1-14)(p_1-15)$, \dots , $p_{15}(p_{15}-1) \dots (p_{15}-14)(p_{15}-15)$ to ensure the p_i have integer values from 0 to 15. The squares cannot be used more than once in this puzzle, so there needs to be equations that ensure that for each p_i, p_j , $p_i - p_j \neq 0$ for all $i \neq j$.

Finally, we create an ideal I generated by the adjacency equations, the domain restrictions, and the uniqueness equations and attempt to create a Gröbner basis of I . Cipra's Puzzle has 3 non-redundant solutions. Our system will give these three solutions with different rotations, flipping the puzzle, and moving rows and columns as distinct solutions.

6. CONCLUSION

Many combinatorics puzzles can be described as a system of polynomial equations using this method. Solving them, on the other hand, proves a problem since these systems are often large, complex, and of multiple variables. We can solve

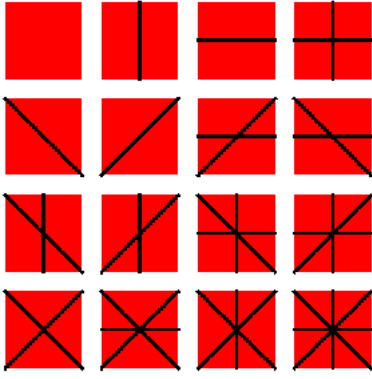


Figure 9: All of the squares in Cipra's Puzzle

these systems using ideals, varieties, and Gröbner bases, but that is dependent on the power of computing available to calculate the Gröbner bases. Lacking high-powered computing, we still are able to calculate a solution using Mathematica's Reduce function iteratively.

7. REFERENCES

- [1] B. Cipra. Problem section. *Math Horizons*, pages 34–35, April 2003.
- [2] D. O. David Cox, John Little. *Ideals, Varieties, and Algorithms*. Springer Science+Business Media, LLC, 233 Spring Street New York, New York 10013, 2007.

APPENDIX

$$\begin{aligned}
 F(g, p, x) = & 1 - \frac{p}{4} - \frac{78433gp}{720} + \frac{1359257g^2p}{5400} - \frac{543211g^3p}{2880} + \frac{542897g^4p}{8640} \\
 & - \frac{27697g^5p}{2880} + \frac{23779g^6p}{43200} + \frac{91p^2}{24} + \frac{230633gp^2}{1440} - \frac{33559g^2p^2}{96} + \frac{292693g^3p^2}{1152} \\
 & - \frac{31967g^4p^2}{384} + \frac{72803g^5p^2}{5760} - \frac{277g^6p^2}{384} + \frac{7p^3}{4} - \frac{28799gp^3}{360} + \frac{704791g^2p^3}{4320} \\
 & - \frac{66073g^3p^3}{576} + \frac{63925g^4p^3}{1728} - \frac{1787g^5p^3}{320} + \frac{2753g^6p^3}{8640} + \frac{5p^4}{24} + \frac{4835gp^4}{288} \\
 & - \frac{45817g^2p^4}{1440} + \frac{24839g^3p^4}{1152} - \frac{7871g^4p^4}{1152} + \frac{1181g^5p^4}{5760} - \frac{337g^6p^4}{720} - \frac{913gp^5}{127x} \\
 & - \frac{48287g^2p^5}{21600} + \frac{233g^3p^5}{160} + \frac{977g^4p^5}{2160} - \frac{97g^5p^5}{1440} + \frac{83g^6p^5}{21600} + \frac{127x}{5} \\
 & - \frac{28079gx}{360} + \frac{922223g^2x}{21600} + \frac{30737g^3x}{2880} - \frac{98947g^4x}{8640} + \frac{7151g^5x}{2880} + \frac{7271g^6x}{43200} \\
 & - \frac{120237px}{400} + \frac{79572127gpx}{43200} - \frac{4281836687g^2px}{1296000} + \frac{187115519g^3px}{86400} \\
 & - \frac{6783517g^4px}{10368} + \frac{8005631g^5px}{86400} - \frac{1613687g^6px}{324000} + \frac{589913p^2x}{1440} \\
 & - \frac{256407851gp^2x}{86400} + \frac{983104573g^2p^2x}{172800} - \frac{26769115g^3p^2x}{6912} + \frac{41522783g^4p^2x}{34560} \\
 & - \frac{29997763g^5p^2x}{172800} + \frac{409453g^6p^2x}{43200} - \frac{32467p^3x}{160} + \frac{5683745gp^3x}{3456} \\
 & - \frac{840906931g^2p^3x}{259200} + \frac{154955873g^3p^3x}{69120} - \frac{29148151g^4p^3x}{41472} + \frac{471781g^5p^3x}{4608} \\
 & - \frac{5833621g^6p^3x}{1036800} + \frac{61123p^4x}{1440} - \frac{6454661gp^4x}{17280} + \frac{5168047g^2p^4x}{6912} \\
 & - \frac{17990137g^3p^4x}{34560} + \frac{5671097g^4p^4x}{34560} - \frac{165965g^5p^4x}{6912} + \frac{2861g^6p^4x}{2160} - \frac{7633p^5x}{2400} \\
 & + \frac{2571797gp^5x}{86400} - \frac{4862633g^2p^5x}{81000} + \frac{1611151g^3p^5x}{38400} - \frac{2751089g^4p^5x}{207360} \\
 & - \frac{672757g^5p^5x}{345600} + \frac{558023g^6p^5x}{5184000} - \frac{100x^2}{144} + \frac{16397gx^2}{960} - \frac{42337g^2x^2}{960} \\
 & - \frac{36887g^3x^2}{1152} + \frac{8389g^4x^2}{384} - \frac{4937g^5x^2}{1152} + \frac{529g^6x^2}{1920} + \frac{761957px^2}{1440} \\
 & - \frac{243677999gpx^2}{86400} + \frac{813783487g^2px^2}{172800} - \frac{10193801g^3px^2}{3456} + \frac{14871571g^4px^2}{17280} \\
 & - \frac{5118853g^5px^2}{43200} + \frac{1075303g^6px^2}{172800} - \frac{435481p^2x^2}{576} + \frac{158395439gp^2x^2}{34560} \\
 & - \frac{345197273g^2p^2x^2}{41472} + \frac{37973983g^3p^2x^2}{6912} - \frac{34540639g^4p^2x^2}{20736} + \\
 & - \frac{1020637g^5p^2x^2}{4320} + \frac{526805g^6p^2x^2}{41472} + \frac{220559p^3x^2}{576} - \frac{17711245gp^3x^2}{6912} + \\
 & - \frac{33521687g^2p^3x^2}{6912} - \frac{90604517g^3p^3x^2}{27648} + \frac{27933157g^4p^3x^2}{27648} - \frac{445903g^5p^3x^2}{3072} \\
 & + \frac{72661g^6p^3x^2}{9216} - \frac{46643p^4x^2}{576} + \frac{20269141gp^4x^2}{34560} - \frac{235842911g^2p^4x^2}{207360} + \\
 & - \frac{5382593g^3p^4x^2}{6912} - \frac{5025221g^4p^4x^2}{20736} + \frac{303031g^5p^4x^2}{8640} - \frac{397379g^6p^4x^2}{207360} +
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{17551p^5x^2}{2880} - \frac{8126137gp^5x^2}{172800} + \frac{3999757g^2p^5x^2}{43200} - \frac{196307g^3p^5x^2}{3072} + \\
 & - \frac{2765047g^4p^5x^2}{138240} - \frac{2010137g^5p^5x^2}{691200} + \frac{36751g^6p^5x^2}{230400} + \frac{127x^3}{8} - \\
 & - \frac{83129gx^3}{1440} + \frac{151709g^2x^3}{8640} + \frac{3967g^3x^3}{192} - \frac{21485g^4x^3}{1728} + \frac{6733g^5x^3}{2880} - \\
 & - \frac{79g^6x^3}{540} - \frac{24401px^3}{80} + \frac{2623603gpx^3}{1728} - \frac{623667331g^2px^3}{259200} + \frac{8372389g^3px^3}{5760} - \\
 & - \frac{1068523g^4px^3}{2592} + \frac{191647g^5px^3}{3456} - \frac{740449g^6px^3}{259200} + \frac{129193p^2x^3}{288} - \\
 & - \frac{8575445gp^2x^3}{3456} + \frac{4966289g^2p^2x^3}{1152} - \frac{12762863g^3p^2x^3}{4608} + \frac{11378459g^4p^2x^3}{13824} - \\
 & - \frac{1588279g^5p^2x^3}{13824} + \frac{84289g^6p^2x^3}{13824} - \frac{133019p^3x^3}{576} + \frac{48151457gp^3x^3}{34560} - \\
 & - \frac{29255033g^2p^3x^3}{11520} + \frac{46434757g^3p^3x^3}{27648} - \frac{4697747g^4p^3x^3}{9216} + \frac{9998867g^5p^3x^3}{138240} - \\
 & - \frac{179213g^6p^3x^3}{46080} + \frac{14213p^4x^3}{288} - \frac{5531453gp^4x^3}{17280} + \frac{3461969g^2p^4x^3}{5760} - \\
 & - \frac{620993g^3p^4x^3}{1536} + \frac{1717075g^4p^4x^3}{13824} - \frac{1229863g^5p^4x^3}{69120} + \frac{66637g^6p^4x^3}{69120} - \\
 & - \frac{10769p^5x^3}{2880} + \frac{889919gp^5x^3}{34560} - \frac{25556413g^2p^5x^3}{518400} + \frac{4633699g^3p^5x^3}{138240} - \\
 & - \frac{860785g^4p^5x^3}{82944} + \frac{68923g^5p^5x^3}{46080} - \frac{168883g^6p^5x^3}{207360} - \frac{19x^4}{6} + \frac{1801gx^4}{144} - \\
 & - \frac{11249g^2x^4}{2880} - \frac{4993g^3x^4}{1152} + \frac{3001g^4x^4}{1152} - \frac{559g^5x^4}{1152} + \frac{173g^6x^4}{5760} + \frac{101947px^4}{1440} - \\
 & - \frac{5847629gpx^4}{17280} + \frac{17793377g^2px^4}{34560} - \frac{1303481g^3px^4}{4320} + \frac{721307g^4px^4}{8640} - \\
 & - \frac{38017g^5px^4}{3456} + \frac{19247g^6px^4}{34560} - \frac{61223p^2x^4}{576} + \frac{19188241gp^2x^4}{34560} - \\
 & - \frac{192770711g^2p^2x^4}{207360} + \frac{1007807g^3p^2x^4}{1728} - \frac{1764913g^4p^2x^4}{10368} + \frac{809449g^5p^2x^4}{34560} - \\
 & - \frac{254729g^6p^2x^4}{207360} + \frac{31909p^3x^4}{576} - \frac{10804819gp^3x^4}{34560} + \frac{19057151g^2p^3x^4}{34560} - \\
 & - \frac{9886411g^3p^3x^4}{27648} + \frac{2957675g^4p^3x^4}{27648} - \frac{691583g^5p^3x^4}{46080} + \frac{36847g^6p^3x^4}{46080} - \\
 & - \frac{6877p^4x^4}{576} + \frac{497695gp^4x^4}{6912} - \frac{27223933g^2p^4x^4}{207360} + \frac{150067g^3p^4x^4}{1728} - \\
 & - \frac{273305g^4p^4x^4}{10368} + \frac{25855g^5p^4x^4}{6912} - \frac{41707g^6p^4x^4}{207360} + \frac{2621p^5x^4}{2880} - \frac{200599gp^5x^4}{34560} + \\
 & + \frac{62341g^2p^5x^4}{5760} - \frac{334171g^3p^5x^4}{46080} + \frac{307153g^4p^5x^4}{138240} - \frac{43907g^5p^5x^4}{138240} + \\
 & + \frac{475g^6p^5x^4}{27648} + \frac{9x^5}{40} - \frac{283gx^5}{288} + \frac{16189g^2x^5}{43200} + \frac{389g^3x^5}{1440} - \frac{191g^4x^5}{1080} + \\
 & - \frac{g^5x^5}{30} - \frac{89g^6x^5}{43200} - \frac{3427px^5}{600} + \frac{575153gp^5x^5}{21600} - \frac{12713507g^2p^5x^5}{324000} + \\
 & - \frac{1937371g^3p^5x^5}{86400} - \frac{157423g^4p^5x^5}{25920} + \frac{22631g^5p^5x^5}{28800} - \frac{25381g^6p^5x^5}{648000} + \frac{6271p^2x^5}{720} - \\
 & - \frac{945911gp^2x^5}{21600} + \frac{4096429g^2p^2x^5}{57600} - \frac{604147g^3p^2x^5}{13824} + \frac{867689g^4p^2x^5}{69120} - \\
 & - \frac{196423g^5p^2x^5}{115200} + \frac{30581g^6p^2x^5}{345600} - \frac{13199p^3x^5}{2880} + \frac{853889gp^3x^5}{34560} - \\
 & - \frac{21967063g^2p^3x^5}{1244263g^3p^3x^5} - \frac{661399g^4p^3x^5}{153079g^5p^3x^5} - \frac{153079g^5p^3x^5}{138240} - \\
 & - \frac{121333g^6p^3x^5}{2073600} + \frac{179p^4x^5}{180} - \frac{49249gp^4x^5}{8640} + \frac{350117g^2p^4x^5}{34560} - \frac{456211g^3p^4x^5}{69120} + \\
 & + \frac{136961g^4p^4x^5}{69120} - \frac{2141g^5p^4x^5}{7680} + \frac{343g^6p^4x^5}{23040} - \frac{1097p^5x^5}{14400} + \frac{79519gp^5x^5}{172800} - \\
 & - \frac{724163g^2p^5x^5}{864000} + \frac{382987g^3p^5x^5}{691200} - \frac{23243g^4p^5x^5}{138240} + \frac{16489g^5p^5x^5}{691200} - \\
 & - \frac{4433g^6p^5x^5}{3456000}
 \end{aligned}$$