Senior Honors Thesis
The Prime Number Theorem and Its Connection with the Riemann Hypothesis

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Abstract: The topic of the paper is the Prime Number Theorem and its connection with the Riemann Hypothesis. The Prime Number Theorem states that $\pi(x) \sim \mathrm{Li}(x)$, where $\pi(x)$ is the number of primes that are less than or equal to x and $\mathrm{Li}(x) = \int_2^x \frac{1}{\ln(t)} dt$. The Riemann Hypothesis provides a good bound for the error and yields an explicit formula for $\pi(x)$ instead of the asymptotic formula. This formula relies on the location of the roots of the Zeta function: $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. In his one and only paper on number theory, Riemann proposed that all non-trivial zeros of the zeta function on complex plane lie on the line $s = ½$, which is known as the critical line. The well-known hypothesis still remains unsolved.

Table of Contents

I. The Prime Number Theorem

1. Introduction

Due to their random behavior and their great importance in number theory and abstract algebra, prime numbers have been an interesting topic for mathematicians to study since the beginning of mathematics. The proof of the infinitude of primes appeared around 300 B.C.E. in Euclid's "Elements". Then, in 1737, Euler proved the divergence of the harmonic series of primes. Towards the end of 18th century, two mathematicians, Gauss and Legendre, working independently, came to the same conjecture about the distribution of primes. Both conjectured that if we let $\pi(x)$ be the number of primes less than or equal to x, then $\pi(x)$ is asymptotically equal to $x/\ln x$:

$$\pi(x) \sim \frac{x}{\ln x}$$

This became known as the Prime Number Theorem.

In 1896, the conjecture was proved independently by Jacques Hadamard and Vallee Pousin at the same time. From that point, the well-stated Prime Number Theorem was officially established. The proof is non-elementary since it uses complex analysis. However, around 1948, an elementary proof emerged as a result of the "collaboration" of two mathematicians, Erdos and Selberg.[1] These proofs will be discussed later in the paper.

2. The Improved Prime Number Theorem

In addition to the logarithm form presented in the Prime Number Theorem above, Gauss, introduced an alternative function to estimate the prime counting function, $\pi(x)$. Using the logarithmic integral function

$$Li(x) = \int_2^x \frac{1}{\ln t} dt,$$

Gauss stated that $\pi(x) \sim Li(x)$.

[1] There was a small debate between the two mathematicians. They did not actually work together, but they used each other's ideas to build a concrete proof of the Prime Number Theorem.

Below is a table taken from Edwards of some values of $\pi(x)$ and its approximations:

x	$\pi(x)$	$x/\ln x$	$Li(x)$
500	95	80.4	101.7
1,000	168	144.7	177.6
2,000	303	263.1	314.8
500,000	41,638	38,102.8	41,606.2
1,000,000	78,498	72,382.4	78,627.5
1,500,000	114,155	105,477.9	114,263.0
2,000,000	148,933	137,848.7	149,054.8
2,500,000	183,072	169,700.9	183,244.9
3,000,000	216,816	201,151.6	216,970.5

Based on the numerical evidence, we can see that the relative error of the approximation of $\pi(x)$ by $Li(x)$ appears to be smaller than the one by $x/\ln x$. So the logarithmic integral seems to give a better asymptotic estimation of $\pi(x)$.

II. Overview of the Proofs of the Prime Number Theorem

1. Riemann's Paper on Number Theory:

In 1859, Riemann published his only paper on number theory titled "On the Number of Prime Numbers less than a Given Quantity". Although the focus of the paper was to estimate the number of primes less than a given number, the ideas presented in the paper have been inspiring many mathematicians in various fields for over 100 years. It is not only the result of the paper, but also his researching methods, such as studying a function in a new region and trying to build an explicit formula as a representation of a function, that became the foundation of later investigation in the mathematical world. In fact, his results given in his paper shed lights on the proof of the Prime Number Theorem about 40 years later.

Among his results was the conjecture of the location of the zeros of the zeta function; it is now known as Riemann Hypothesis. Although it remains unsolved, even until now, mathematicians have been attempting to prove the hypothesis with the belief that the proof of the Riemann Hypothesis will reveal innovative techniques for further research.

2. Introduction of the Riemann Zeta function

a. The Zeta function:

Starting with Euler's formula for the sum of the reciprocals $\sum_{n=1}^{\infty} n^{-s}$, where s is integer and n ranges over all positive integers, Riemann considered s as a complex variable and studied the function on the new complex plane.

Using the factorial function and contour integration, Riemann derived a formula for $\sum_{n=1}^{\infty} n^{-s}$ that "remains valid for all s"[2]

$$\zeta(s) = \frac{\Pi(-s)}{2\pi i} \int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \cdot \frac{dx}{x} \qquad (1)$$

This contour integral implies the path of integration; it starts at $+\infty$, goes to the left along the positive x-axis, circles around the origin in the counterclockwise direction, and moves back to $+\infty$ in the positive x-axis.

Based on the formula (1) of $\zeta(s)$, for Re(s) > 1, $\zeta(s)$ is equal to Dirichlet's function[3]

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \sum_{n=1}^{\infty} n^{-s}. \qquad (2)$$

Furthermore, as Re(s) > 1, Riemann observed that $\zeta(s)$ can be expressed as Euler product formula[4]

$$\zeta(s) = \prod_p \frac{1}{(1 - p^{-s})} \qquad (3)$$

where p ranges over all primes (p = 2, 3, 5, 7, ...).

The function $\zeta(s)$ is now known as the Riemann Zeta function due to Riemann's original contributions for the function.

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[2] The factorial function is defined as: $s! = \Pi(s) = \int_0^{\infty} e^{-x} x^s \, dx$. The notation $\Pi(s)$ was introduced by Gauss.

[3] Dirichlet's function is different from Euler's because its domain is real numbers that are greater than 1 rather than integers only.

[4] Because Euler product formula ranges over all prime numbers p, the equation in (3) shows the primary connection between $\zeta(s)$ and prime numbers.

b. Characters of the Zeta function:

In his paper, after defining the zeta function, Riemann analyzed its properties. In this section, a few important properties and sketches of their proofs are shown.

Property 1: $\zeta(s)$ is analytic and defined to the entire complex plane except for a simple pole at $s = 1$.

Proof:

Because e^x grows much faster than x^s when $x \to \infty$, the integral in (1) converges for all values of s. And since convergence is uniform on compact domains, the integral defines a complex analytic function. Hence, the overall function, $\zeta(s)$, is defined and analytic on the entire complex plane except the possible points where s = 1, 2, 3, …, where $\Pi(-s)$ has poles.[5]

At s = 2, 3, 4, the formula (2) shows that $\zeta(s)$ has no pole. In facts, at these points, the function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ converges. And at $s = 1$, since we already know that the harmonic series $\sum_{n=1}^{\infty} n^{-s}$ diverges, $\zeta(s)$ has a simple pole at $s = 1$.

In short, $\zeta(s)$ is shown to be analytic and defined to the entire complex plane except for a simple pole at $s = 1$. Q. E. D.

Property 2: The relationship between $\zeta(s)$ and $\zeta(1-s)$ is established through the formula:

$$\zeta(s) = \Pi(-s)(2\pi)^{s-1} 2\sin\left(\frac{s\pi}{2}\right)\zeta(1-s), \zeta(1-s), \qquad (4)^6$$

which is known as the ***functional equation of the zeta function.***[7]

[5] In complex analysis, a pole of a function is basically a point where the function approaches infinity as the variable approaches that point.

[6] Since for integer k, sin(k π) = 0, the formula suggests that the roots of $\zeta(s)$ includes all negative even integers s = - 2n. However, positive even integers can't be the roots of $\zeta(s)$ because when s = 2n, $\sin(n\pi)\Pi(s)$ is regular, which means $\zeta(s)$ becomes a convergent series and it is different from 0. The formula (4) also indicates that $\zeta(s)$ has no odd negative roots, which helps the process of locating the zeros of $\zeta(s)$.

[7] Riemann derived this formula using Cauchy's theorem and Cauchy integral formula.

Proof:

By making use of the basic properties of the factorial function, the formula (4) can be rewritten as

$$\Pi\left(\frac{s}{2}-1\right)\pi^{-s/2}\zeta(s)=\Pi\left(\frac{1-s}{2}-1\right)\pi^{-(1-s)/2}\zeta(1-s). \qquad (5)$$

Since the value of the function on the left-hand side remains unchanged when s is replaced by $(1-s)$, the formula in (4) is referred as the functional equation of the zeta function.

Q. E. D.

Note: In his paper, Riemann also showed another proof of the functional equation. He borrowed the functional equation of the theta function, $\frac{1+2\psi(x)}{1+2\psi(\frac{1}{x})}=\frac{1}{\sqrt{x}}$ where $\psi(x)=\sum_{n=1}^{\infty}e^{-n^2\pi x}$, from Jacobi. Then he derived another symmetrical formula:

$$\Pi\left(\frac{s}{2}-1\right)\pi^{-s/2}\zeta(s)=\int_1^{\infty}\psi(x)[x^{s/2}+x^{1-s/2}]\frac{dx}{x}-\frac{1}{s(1-s)}. \qquad (6)$$

Because the right-hand side of formula stays unchanged when $(1-s)$ is substituted for s and vice versa, it proved the functional equation of the zeta function.

The symmetrical form of the functional equation leads to the introduction of a new function. Riemann defined the function

$$\xi(s)=\Pi\left(\frac{s}{2}\right)(s-1)\pi^{-s/2}\zeta(s). \qquad (7)$$

By this definition, $\xi(s)$ is an analytic function of s, which is defined for all values of s. In other words, $\xi(s)$ is an entire function. Furthermore, the function $\xi(s)$ also verifies the functional equation of $\zeta(s)$; from (5) and (7), we can get $\xi(s)=\xi(1-s)$, which is equivalent to the functional equation of zeta function.

Property 3: The zeros of $\xi(s)$ have their real parts between 0 and 1.[8]

Proof:

Because in the formula of $\xi(s)$ at (7), other factors beside $\zeta(s)$ has only one simple zero at $s = 1$, the roots of $\xi(s)$ are the same as the roots of $\zeta(s)$. Then as it is proved in Properties 2 that $\zeta(s)$ is zero-free on the half-plane where Re(s) >1, $\xi(s)$ has no root on that half-plane either.

Moreover, the equation $\xi(s) = \xi(1-s)$ implies that $(1 - \rho)$ is a root of $\xi(s)$ if and only if ρ is a root of $\xi(s)$. Hence, since it is shown that $\xi(s)$ has no root on the half plane Re(s) >1, $\xi(s)$ does not have any root on the half-plane Re(s) < 0. Therefore, all the roots of $\xi(s)$, if existing, have to lie in the strip $0 \leq \text{Re}(s) \leq 1$. Q. E. D.

3. Complex Analysis Proof by Hadamard and Valle-Poussin:

In 1896, the Prime Number Theorem was proved for the first time by two mathematicians, Jacques Hadamard and Charles Jean de la Vallée-Poussin. Although they worked independently, their proofs were similar and came out to the public around the same time. Therefore, they both got credited for proving the Prime Number Theorem.

Due to its historical significance, a sketch of the proof is laid out in this section. The proof followed the ideas in Riemann's paper and made great use of the zeta function and its properties discovered by Riemann and other mathematicians over time. It included two main parts. The first one is related to the distribution of the zeros of the Riemann zeta function and the second one is the derivation of the Prime Number Theorem using the result from the first part and a few functional transformations.

a. First step of the proof of the Prime Number Theorem:

In his paper, Hadamard claimed that $\zeta(s) \neq 0$ when Re(s) = 1 and his proof is shown below.[9]

[8] This fact helps to locate the roots of the zeta function.
[9] Although both proofs are intricate, the one by Hadamard is simpler and it is why I chose to re-demonstrate it here.

Let $s = \sigma + it$ and consider $s > 1$. Then

$$\ln|\zeta(s)| = \mathrm{Re}(\ln \zeta(s)) = \sum_{n=2}^{\infty} c_n n^{-s} \cos(t \ln t),$$

where $c_n = \begin{cases} \frac{1}{m} \text{ if } n = p^m, \ p \text{ is prime} \\ 0 \qquad\qquad \text{otherwise} \end{cases}.$

It follows that

$$\ln\left|\zeta(\sigma)^3 \zeta(\sigma+it)^4 \zeta(\sigma+i2t)\right| = \sum_{n=2}^{\infty} c_n n^{-s}(3 + 4\cos(t\ln n) + \cos(2t\ln n)).$$

Because 3 + 4 cos t + cos 2t = 2 + 4 cos t + 2(cos t)^2 = 2(1 + cos t)^2 ≥ 0,

$$\ln\left|\zeta(\sigma)^3 \zeta(\sigma+it)^4 \zeta(\sigma+i2t)\right| \geq 0.$$

Then $\zeta(\sigma)^3 \zeta(\sigma+it)^4 \zeta(\sigma+i2t) \geq 1$.

Thus, $((\sigma-1)\zeta(\sigma))^3 \left|\frac{\zeta(\sigma+it)^4}{\sigma-1}\right| |\zeta(\sigma+i2t)| \geq \frac{1}{\sigma-1}$ (8)

for $\sigma > 1$ and for all values of t.

Since $\zeta(s)$ has a simple pole at $s = 1$, we have

$$\lim_{\sigma \to 1} (\sigma - 1)\zeta(s) = 1.$$

Suppose $t \neq 0$ and assume $\zeta(1+it) = 0$.

Then we would have

$$\lim_{\sigma \to 1} \frac{\zeta(\sigma+it)}{\sigma-1} = \zeta'(1+it).$$

In addition, $\lim_{\sigma \to 1} \frac{1}{\sigma-1} = \infty$. Hence, (8) implies $\lim_{\sigma \to 1} |\zeta(\sigma+2it)| = \infty$. And because $\zeta(s)$ has only one simple pole at $s = 1$, it indicates that t has to be 0, which contradicts to the assumption that $t \neq 0$. Therefore, $\zeta(1+it) \neq 0$. Q. E. D.

b. Second step of the proof of the Prime Number Theorem:

- 1: Show that $\Psi(x) \sim x$.[10]

Since no one had been able to prove the theorem directly from $\pi(x)$ till his time, Hadamard decided to approach the theorem indirectly. He used another function that behaves similar to $\pi(x)$ but is easier to estimate in his proof. He introduced the step function $\Psi(x)$, which starts at 0 and has a jump of $\ln p$ at each prime power p^n.[11] So the formula of $\Psi(x)$ is

$$\Psi(x) = \sum_{p^n < x} \ln p \qquad (9)$$

By evaluating the definite integral $\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left[-\frac{\zeta'(s)}{\zeta(s)}\right]\frac{x^s ds}{s}$, Hadamard obtained a representation for $\Psi(x)$:

$$\Psi(x) = x - \sum \frac{x^\rho}{\rho} + \sum \frac{x^{-2n}}{2n} - \frac{\zeta'(0)}{\zeta(0)} \qquad (10)$$

where $x > 1$ and ρ ranges over the zeros of the Riemann zeta function.

Then $$\int_0^x \Psi(t)dt = \frac{x^2}{2} - \sum_\rho \frac{x^{\rho+1}}{\rho(\rho+1)} - \sum_n \frac{x^{-2n+1}}{2n(2n+1)} - \frac{\zeta'(0)}{\zeta(0)}x + \frac{\zeta'(-1)}{\zeta(-1)}. \qquad (11)$$

Consider $\dfrac{\int_0^x \Psi(t)\,dt - x^2/2}{x^2/2}$. By (11), it will be equal to $2\sum_\rho \frac{x^{\rho-1}}{\rho(\rho+1)}$ plus "some other terms", which will go to 0 as x goes to infinity.

Since $\sum_\rho \frac{1}{\rho(\rho+1)}$ converges absolutely and $|x^{\rho-1}| \le 1$, the series $\sum_\rho \frac{x^{\rho-1}}{\rho(\rho+1)}$ converges uniformly and the limit as x goes to infinity can be evaluated termwise.

Moreover, since $\lim_{x\to\infty} \frac{x^{\rho-1}}{\rho(\rho+1)} = 0$ due to the fact that $\mathrm{Re}(\rho) < 1$ (from step 1), we have

$$\lim_{x\to\infty} 2\sum \frac{x^{\rho-1}}{\rho(\rho+1)} = 0.$$

Then it follows that $\int_0^x \Psi(t)\,dt \sim \frac{x^2}{2}$ and we will be able to show $\Psi(x) \sim x$. Q. E. D.

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[10] This $\Psi(x)$ is different from the exponential function $\psi(x)$ in 6.

[11] The function $\Psi(x)$ was first considered by Chebyshev.

- 2: Deduce the Prime Number Theorem[12]

Since the approximation $\Psi(x) \sim x$ is shown above, proving the Prime Number Theorem is equivalent to deducing $\pi(x)$ from $\Psi(x)$. The technique described below is the one that was used by Chebyshev in 1850.

We need to define a new prime-counting function $\theta(x)$. $\theta(x)$ represents the sum of the logarithms of all the primes p less than x.[13] Then $\theta(x)$ and $\Psi(x)$ are connected through the formula

$$\Psi(x) = \theta(x) + \theta(x^{1/2}) + \theta(x^{1/3}) + \theta(x^{1/4}) + \ldots$$

We notice that for n sufficiently large, $x^{1/n} < 2$, then $\theta(x) = 0$. Thus, the series above is a finite sum and there are at most $\frac{\ln x}{\ln 2}$ nonzero terms and

$$\theta(x) < \Psi(x) < \theta(x) + \theta\left(x^{1/2}\right)\frac{\ln x}{\ln 2}.$$

Then
$$\Psi(x) - \theta\left(x^{\frac{1}{2}}\right)\frac{\ln x}{\ln 2} < \theta(x) < \Psi(x).$$

And
$$\frac{\Psi(x)}{x} - \frac{\theta(x^{1/2})}{x}\frac{\ln x}{\ln 2} < \frac{\theta(x)}{x} < \frac{\Psi(x)}{x}. \qquad (12)$$

Since $\Psi(x) \sim$ x and θ(x) $< \Psi(x)$, $\lim_{x\to\infty}\frac{\theta(x)}{x^{1+\varepsilon}} = 0$. Hence

$$\frac{\theta(x^{1/2})}{x}\ln x = \left[\frac{\theta(x^{1/2})}{(x^{1/2})^{1+\varepsilon}}\right]\left[\frac{\ln x}{(x^{1/2})^{1-\varepsilon}}\right] \to 0 \text{ as } x \to \infty \qquad (13)$$

From the inequalities in (12), the limit in (13), and the result that $\Psi(x) \sim x$, we conclude

$$\theta(x) \sim x.$$

Given $\varepsilon > 0$ and let X be a number such that $\left|\frac{\theta(x)}{x} - 1\right| \le \varepsilon$ whenever $x \ge$ X. Then for $y > x \ge$ X, we get

$$\pi(y) - \pi(x) = \int_x^y \frac{d\theta(t)}{\ln t} = \left[\frac{\theta(t)}{\ln t}\right]_x^y + \int_x^y \frac{d\theta(t)}{(\ln t)^2 t}$$

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[12] This deduction is taken from Edward's book

[13] If x itself is a prime, $\theta(x) = \frac{1}{2}[\theta(x+\varepsilon) + \theta(x-\varepsilon)]$

is at most

$$\frac{(1+\varepsilon)y}{\ln y}-\frac{(1-\varepsilon)x}{\ln x}+\int_x^y\frac{(1+\varepsilon)tdt}{(\ln t)^2 t}=2\varepsilon\frac{x}{\ln x}+(1+\varepsilon)\left\{\left[\frac{t}{\ln t}\right]_x^y+\int_x^y\frac{tdt}{(\ln t)^2 t}\right\}$$

$$=2\varepsilon\frac{x}{\ln x}+(1+\varepsilon)\left\{\int_x^y\frac{dt}{\ln t}\right\}=2\varepsilon\frac{x}{\ln x}+(1+\varepsilon)[Li(y)-Li(x)]$$

and at least equal to $-2\varepsilon\frac{x}{\ln x}+(1-\varepsilon)[Li(y)-Li(x)]$.

Therefore, for a fixed x, $\frac{\pi(y)}{Li(y)}$ is at most

$$\frac{2\varepsilon\frac{x}{\ln x}+(1+\varepsilon)[Li(y)-Li(x)]+\pi(x)}{Li(y)}=1+\varepsilon+\frac{2\varepsilon\frac{x}{\ln x}+(1+\varepsilon)Li(x)+\pi(x)}{Li(y)}\le 1+2\varepsilon$$

and is at least $1-2\varepsilon$ for sufficiently large y. Because ε is an arbitrary number, this implies $\frac{\pi(y)}{Li(y)}\to 1$, or equivalently, $\pi(y)\sim Li(y)$. The Prime Number Theorem is proved.

Q. E. D.

c. ***Other proofs of the Prime Number Theorem:***

After the first proof of the Prime Number Theorem by Hadamard and Poussin, more proofs came out; some of them were shorter, but they all involve difficult complex analysis. In 1949, Atle Selberg and Paul Erdős found the first elementary proof. Since the proof avoided the use of complex analysis, it was considered "elementary". However, it was less natural and less intuitive than the proof via Riemann's zeta function while still remaining quite elaborate and not easy to comprehend. Therefore, the analytic proofs were still preferred.

In 1980, nevertheless, a very simple proof of the Prime Number Theorem was given by D. J. Newman. Modifying Tauberian argument, Newman constructed a proof with a simple and concise structure that does not require much background of complex analysis; he barely used any heavy machinery, but the basic Cauchy's theorem. Hence, this proof was welcome by a larger audience and suitable for readers who would like to learn about the proof of the Prime Number Theorem, without a solid background in complex analysis.

III. Relationship between the Prime Number Theorem and the Riemann Hypothesis:

1. The Riemann Hypothesis:

In his paper on number theory, while studying the zeta function and trying to find a good estimate of $\pi(x)$, Riemann proposed that it is likely that all non-trivial zeros of the Riemann zeta function lie on the same line where the real part is ½. This statement is now known as the famous Riemann Hypothesis. It remains as one of seven unsolved millennium prize problems of Clay Mathematics Institute; 1 million dollars is allocated to give to the person who can solve the problem. It is believed that a proof of the Riemann Hypothesis will shed light on the mystery of the distribution of prime numbers. However, the failure of the Riemann Hypothesis, if found, would create chaos to the distribution of prime numbers. Therefore, due to the significance of its validity, the Riemann Hypothesis is considered as one of the most important unsolved problems of mathematics.

2. The Relationship between the Theorem and the Hypothesis:

a. The explicit formula for $\pi(x)$:

We already know that prime numbers do not follow any pattern and seem to behave randomly, so it is quite difficult to predict their value given their location in the sequence of primes. Fortunately, Riemann observed that their distribution turns out to be closely related to the behavior of his zeta function, $\zeta(s)$. Specifically, his goal for his paper is to find an explicit analytic expression for $\pi(x)$ so that we can get the exact value of $\pi(x)$ instead of the asymptotic approximation in the Prime Number Theorem. Assuming the Riemann Hypothesis is true, he achieved his goal and his construction of the explicit formula is shown below.

- Introduction of the Prime-Jumping Function $J(x)$:[14]

Taking the log of both sides of Euler product formula in (3), we get

$$\ln \zeta(s) = \sum_p \left[\sum_n \left(\frac{1}{n}\right) p^{-ns} \right] \qquad (\mathrm{Re}(s) > 1) \qquad (14)^{15}$$

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[14] Riemann originally denoted the function f(x). However, we use f(x) for any general function. Thus, to avoid confusion, Edwards came up with the new notation $J(x)$ and this paper will follow that.
[15] We also need to use the Taylor expansion of $\ln(1-x)$, which is equal to $-x - (1/2)x^2 - (1/3)x^3 - \ldots$ to derive that form.

In order to rewrite this equation under integral form, Riemann defined a new function $J(x)$ that I would like to refer as the Prime-Jumping Function. $J(x)$ is a function that starts at 0 when $x = 0$ and increases by a jump of at at primes p, by a jump of 1/2 at prime squares p^2, by a jump of 1/3 at prime cubes p^3, and so on. $J(x)$ can be expressed in the form

$$J(x) = \frac{1}{2}\left[\sum_{p^n < x} \frac{1}{n} + \sum_{p^n \le x} \frac{1}{n}\right]$$

Then $$\ln \zeta(s) = s\int_0^\infty J(x)x^{-s-1}\,dx$$

And $$\frac{\ln \zeta(s)}{s} = \int_0^\infty J(x)x^{-s-1}\,dx \qquad (15)$$

Applying Fourier inversion[16] to (15), Riemann concluded

$$J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \ln \zeta(s) x^s \frac{ds}{s} \qquad \text{with a > 1} \qquad (16)$$

$$\text{and } J(x) = -\frac{1}{2\pi i}\frac{1}{\ln x}\int_{a-i\infty}^{a+i\infty} \frac{d}{ds}\left[\frac{\ln \zeta(s)}{s}\right] x^s ds \qquad (17)^{17}$$

This is the representation of $J(x)$ in terms of $\zeta(s)$. The next step is to derive a formula that is easy to evaluate for $J(x)$.

- Riemann's Formula for $J(x)$:

First, Riemann expanded ln $\zeta(s)$ using 2 identity formulas of $\xi(s)$:

$$\xi(s) = \Pi\left(\frac{s}{2}\right)\pi^{-s/2}(s-1)\zeta(s)$$

$$\text{and} \qquad \xi(s) = \xi(0)\Pi_\rho\left(1-\frac{s}{\rho}\right)$$

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[16] To transform $J(x)$, Riemann used the Fourier theorem, which states that a function f(x) can be written as a superposition of exponentials

$$f(x) = \int_{-\infty}^{\infty} g(y)e^{iyx}dy$$

if and only if the coefficients g(y) is defined as

$$g(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(z)e^{-izy}dz.$$

In our case, $z = \ln x$; $f(x) = 2\pi J(e^x)e^{-ax}$; and $g(y) = \frac{\ln \zeta(s)}{s}$ where y = Im(s).

[17] (17) is derived from (16) by integration by parts.

Combining these two, we get:

$$\ln\zeta(s) = \ln\xi(s) - \ln\Pi\left(\frac{s}{2}\right) + \frac{s}{2}\ln\pi - \ln(s-1)$$

$$= \ln\xi(0) + \sum_{\rho}\ln\left(1-\frac{s}{\rho}\right) - \ln\Pi\left(\frac{s}{2}\right) + \frac{s}{2}\ln\pi - \ln(s-1) \qquad (18)$$

Substituting (18) back to (17), we obtain an explicit formula for J(x), which is the sum of five terms. The value of J(x) depends on how the five terms are evaluated.

First term: $-\frac{1}{2\pi i}\frac{1}{\ln x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\ln\xi(0)}{s}\right]x^s ds = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\ln\xi(0)}{s}x^s\, ds = \ln\xi(0)$.

Because $\xi(0) = \Pi(0)\pi^{-s}(0-1)\zeta(0) = -\zeta(0) = \frac{1}{2}, \ln\xi(0) = -\ln 2$.

And this is the numerical value of the first term in the Riemann's formula for J(x).

Second term: $-\frac{1}{2\pi i}\frac{1}{\ln x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\sum_{\rho}\ln\left(1-\frac{s}{\rho}\right)}{s}\right]x^s ds.$

Using contour integral, we can transform this term to

$$-\sum_{\operatorname{Im}\rho>0}[Li(x^{\rho}) + Li(x^{1-\rho})]$$

This expression reveals the initial connection between $J(x)$ and $Li(x)$ and equivalently, the zeta function $\zeta(s)$ and the Prime Number Theorem.[18]

Third term: $-\frac{1}{2\pi i}\frac{1}{\ln x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{-\ln\Pi\left(\frac{s}{2}\right)}{s}\right]x^s ds = \frac{1}{2\pi i}\frac{1}{\ln x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\ln\Pi\left(\frac{s}{2}\right)}{s}\right] \qquad (19)$

[18] Riemann used a different from of the log integral function. He defined

$$Li(x) = \int_0^x \frac{1}{\ln(t)}dt$$

I will use this formula to derive the explicit formula of π(x), specifically in part a and b only. In other sections, I still follow Gauss's notation.

Using $\ln \Pi\left(\frac{s}{2}\right)=\sum_{n=1}^{\infty}\left[-\ln\left(1+\frac{s}{2n}\right)+\frac{s}{2}\ln\left(1+\frac{1}{n}\right)\right]$, we can conclude that (19) is equal to

$$\int_x^{\infty}\frac{dt}{t(t^2-1)\ln t}.$$ [19]

Forth term: $$-\frac{1}{2\pi i}\frac{1}{\ln x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\frac{s}{2}\ln\pi}{s}\right]x^s ds=-\frac{1}{2\pi i}\frac{1}{\ln x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\ln\pi}{2}\right]x^s ds$$

$$=-\frac{1}{2\pi i}\frac{1}{\ln x}\int_{a-i\infty}^{a+i\infty}0x^s\,ds=0.$$ [20]

Fifth term: $$-\frac{1}{2\pi i}\frac{1}{\ln x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{-\ln(s-1)}{s}\right]x^s ds=\frac{1}{2\pi i}\frac{1}{\ln x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\ln(s-1)}{s}\right]$$

Riemann proved that for x > 1, the value of this definite integral is the logarithmic integral

$$\frac{1}{2\pi i}\frac{1}{\ln x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\ln(s-1)}{s}\right]=\lim_{\varepsilon\downarrow 0}\left[\int_0^{1-\varepsilon}\frac{dt}{\ln t}+\int_{1+\varepsilon}^{x}\frac{dt}{\ln t}\right]=Li(x).$$

The proof of the equation is related to Fourier inversion and contour integration. Details can be found in Edward's book. This term is also considered as the principal term of *J*(*x*).

In short, putting pieces together, we obtain an analytic formula for *J(x)*

$$J(x)=Li(x)-\sum_{\operatorname{Im}\rho>0}\left[Li(x^{\rho})+Li(x^{1-\rho})\right]+\int_0^{\infty}\frac{dt}{t(t^2-1)\ln t}-\ln 2 \qquad (20)$$

where ρ runs over the complex roots of the zeta function and $x > 1$.

In fact, this formula is the main result of Riemann's paper. Since it uses zeros of the zeta function to evaluate the Prime-Jumping function *J*(*x*), it is the bridge that connects the Riemann Hypothesis and the Prime Number Theorem.

[19] The argument requires the condition that the termwise integration is valid for the equation. The proof can be found in Edwards' book.

[20] The differentiation inside is equal to 0 because we differentiate with respect to s and the term s already gets cancelled from the previous step.

- The Expression of $\pi(x)$ in terms of $J(x)$

Based on the definition of $J(x)$, Riemann found a relationship between $\pi(x)$ and $J(x)$

$$J(x) = \pi(x) + \frac{1}{2}\pi(\sqrt{x}) + \frac{1}{3}\pi(\sqrt[3]{x}) + \frac{1}{4}\pi(\sqrt[4]{x}) + \ldots = \sum_{i=1}^{\infty} \frac{1}{i}\pi(\sqrt[i]{x}).$$

Then, by the Mobius Inversion[21], he inverted the order of the equation and got

$$\pi(x) = J(x) - \frac{1}{2}J(\sqrt{x}) - \frac{1}{3}J(\sqrt[3]{x}) - \ldots \qquad (21)$$

The first sum is actually finite for each given x since $x^{1/n} < 2$ for a sufficiently big value of n, which leads to $\pi(x^{1/n}) = 0$. Then it follows that the second series, the representation of $\pi(x)$ in terms of $J(x)$, is finite also.

- The Explicit Formula of $\pi(x)$:

Substituting (20) to (21), Riemann obtained an explicit formula of the Prime-Counting function $\pi(x)$ as he desired. This formula includes 3 types of terms:

- The stable terms, which do not grow as x increases: They are the last two terms in (20).
- The terms that grow steadily as x increases: They consist of the terms $Li(x)$.
- The terms that grow but oscillate in sign as x increases: They come from the sum of $Li(x^{\rho})$ over the imaginary part of the roots of zeta function. Because of the oscillation of these terms, it hinders the calculation of the explicit formula of $\pi(x)$.

In addition, the fact that $\pi(x)$ is expressed in terms of a sum over the zeros of the Riemann zeta function implies that the magnitude of the oscillations of primes around their expected positions is controlled by the zeros of the zeta function.

[21] The Mobius inversion formula interchanges the positions of two functions in an equation by following the rule $g(x) = \sum_{n=1}^{\infty} f(nx) \Leftrightarrow f(x) = \sum_{n=1}^{\infty} \mu(n)g(nx)$ provided that both $\sum f(nx)$ and $\sum g(nx)$ converge absolutely. $\mu(n)$ is the Mobius function that is defined in the next section.

b. *Riemann's Approximation of $\pi(x)$*

Because the oscillating terms $\sum_{\operatorname{Im}\rho>0}\left[Li(x^{\rho})+Li(x^{1-\rho})\right]$ are hard to evaluate and some of the terms cancel each other due to opposite signs, Riemann dropped these terms from the formula and suggested an approximation of $\pi(x)$:

$$\pi(x)\sim Li(x)-\frac{1}{2}Li\left(x^{1/2}\right)-\frac{1}{3}Li\left(x^{1/3}\right)-\frac{1}{5}Li\left(x^{1/5}\right)+\frac{1}{6}Li\left(x^{1/6}\right)-\frac{1}{7}Li\left(x^{1/7}\right)+\ldots$$

or in a shorter form, $\pi(x)\sim Li(x)+\sum_{n=2}^{\infty}\frac{\mu(n)}{n}Li\left(x^{1/n}\right)$ (22)

where $\mu(n)$ is the Mobius function

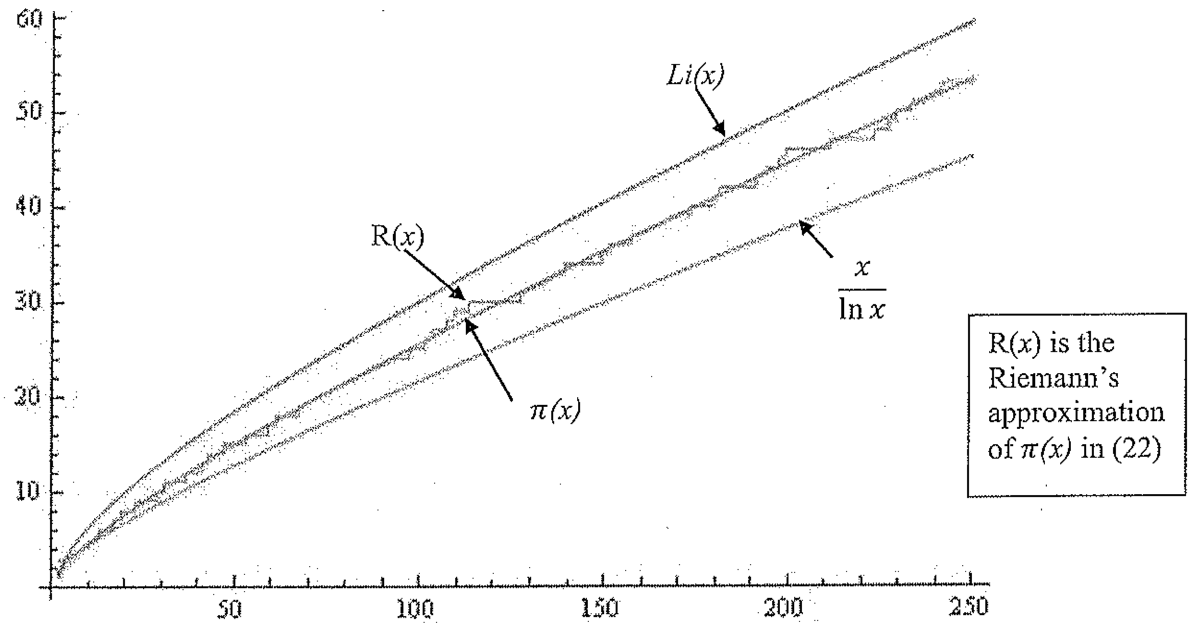
$$\mu(n)=\begin{cases}1 \text{ if } n \text{ is a square}-\text{free integer with an even number of prime factors}\\ -1 \text{ if } n \text{ is a square}-\text{free integer with an odd number of prime factors}\\ 0 \quad \text{if } n \text{ is not square}-\text{free}\end{cases}$$

This formula is called the Riemann's formula for $\pi(x)$.

Furthermore, in fact, the first two terms in (22) is the approximation of $\pi(x)$ in the Improved Prime Number Theorem

$$\pi(x)\sim Li(x)-\frac{1}{2}Li\left(x^{1/2}\right)=Li(x)-Li(2)=\int_{2}^{x}\frac{1}{\ln t}dt.$$

Therefore, it indicates that the formula in (22) gives a closer estimate of $\pi(x)$. The graph below includes all approximations of $\pi(x)$, which makes it clear how distinctively good and natural the approximation of $\pi(x)$ in the formula (22) is.



The efficiency of Riemann's formula for $\pi(x)$ is also illustrated by empirical data provided by Lehmer as follows:

x	Riemann's error	Gauss's error
1,000,000	30	130
2,000,000	- 9	122
3,000,000	0	155
4,000,000	33	206
5,000,000	- 64	125
6,000,000	24	228
7,000,000	- 38	179
8,000,000	- 6	223
9,000,000	- 53	187
10,000,000	88	339

c. The error term of the Prime Number Theorem:

In his paper, Riemann also set up the connection between the relative error in the asymptotic approximation of $\pi(x)$ and the distribution of the complex zeros of the Riemann zeta function. Assuming his hypothesis about the nontrivial zeros of the zeta function is true, Riemann was able to give an exact analytical formula for the error of the approximation of $\pi(x)$

$$\pi(x) - \sum_{n=1}^{N} \frac{\mu(n)}{n} Li\left(x^{1/n}\right) = \sum_{n=1}^{N} \sum_{\rho} Li\left(x^{\rho/n}\right) + \text{"some lesser terms".}$$

Moreover, it is stated that the Riemann Hypothesis is equivalent to a much better error bound in the Prime Number Theorem. In fact, in 1901, assuming the validity of the Riemann Hypothesis, von Koch was able to use Riemann's formula of $\pi(x)$ and successfully proved this statement; additionally, he showed that

$$\pi(x) - Li(x) = O\left(\sqrt{x}\ln x\right).$$

It implies that the approximation of $Li(x)$ to $\pi(x)$ is square-root accurate. And this error bound is considered as the best possible one that can be obtained.

Furthermore, although Riemann did not prove the Prime Number Theorem in his paper, the Prime Number Theorem can be implicitly derived from his argument. Riemann found out that

$$\pi(x) = Li(x) + O\left(x^{1/2}\right).$$

This implies $\frac{\pi(x)}{Li(x)} = 1 + O\left(x^{-1/2}\ln x\right) = 1 + O(1)$, which actually leads to the Prime Number Theorem.

IV. The Verification of the Riemann Hypothesis:

As it was mentioned earlier in the paper, $\zeta(s)$ is nonzero throughout the half plane $\mathrm{Re}(s) > 1$. In addition, it was shown that the zeros of $\zeta(s)$ are classified into 2 types: the trivial zeros, which are located at each even negative integer, and the nontrivial zeros, which stay strictly inside the strip $0 < \mathrm{Re}(s) < 1$.[22] The Riemann Hypothesis concerns about the non-trivial zeros and asserts that all non-trivial zeros should lie on the same line called the **critical line,** ½ + *it*, where t is a real number and *i* is the imaginary unit.

1. Location of trivial zeros

Since the function $x(e^x - 1)^{-1}$ is analytic near $x = 0$, it can be expanded as a power series

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!} \qquad (23)$$

where the coefficients B_b are called Bernoulli numbers. It is noticed that the odd Bernoulli numbers B_{2n+1} are all 0 except the first and there is no simple formula for the even Bernoulli numbers B_{2n} but they can still be found successively.

When $s = -n$ ($n = 0, 1, 2, \ldots$), we can substitute (23) into the defining formula of zeta function in (1) and obtain

$$\zeta(-n) = \frac{\Pi(n)}{2\pi i} \int_{-\infty}^{+\infty} \frac{(-x)^n}{e^x - 1} \frac{dx}{x}$$

$$= \frac{\Pi(n)}{2\pi i} \int_{|x|=\delta} \sum_m \frac{B_m x^m}{m!} \frac{(-x)^n}{x} \frac{dx}{x} = \sum_m \Pi(n) \frac{B_m}{m!} (-1)^n \frac{1}{2\pi} \int_0^{2\pi} x^{m-n-1} \, d\theta$$

$$= n! \frac{B_{n+1}}{(n+1)!} (-1)^n = (-1)^n \frac{B_{n+1}}{n+1}.$$

Then $\zeta(-2n) = (-1)^{2n} \frac{B_{2n+1}}{2n+1}$. Thus, $\zeta(-2n) = 0$ for all $n = 0, 1, 2, \ldots$

Therefore, all even negative integers are roots of the zeta function and they are classified as "trivial zeros".[23]

2. Methods of locating the nontrivial roots:

The nontrivial zeros of the Riemann zeta function are proposed to lie on the same line where the real part is ½. Because no proof for this hypothesis has been found, mathematicians have been analyzing the zeta function numerically and a lot of nontrivial zeros have been located

[22] The strip is known as the critical strip.
[23] They are called trivial because they do not carry significant mathematical meaning.

on the critical line while no counterexample—nontrivial zeros falling off the critical line has yet been detected. This section will show two techniques of finding the nontrivial roots manually.[24]

a. Euler-Maclaurin Summation:

$\xi(s)$ is real-valued on the critical line since with $s = 1/2 + it$, we can express $\xi(s)$ as follows

$$\xi\left(\frac{1}{2}+it\right)=4\int_{1}^{\infty}\frac{d\left[x^{3/2}\psi'(x)\right]}{dx}x^{-1/4}\cos\left(\frac{t}{2}\ln x\right)dx$$

Thus, this implies that we can approximate the zeros by locating the interval where the function changes sign.

$$\xi\left(\frac{1}{2}+it\right)=\frac{s}{2}\Pi\left(\frac{s}{2}-1\right)\pi^{-s/2}(s-1)\zeta(s)=e^{\ln\pi[s/2-1]}\pi^{-s/2}\frac{s(s-1)}{2}\zeta(s)$$

$$=\left[e^{\mathrm{Re}\ln\pi[s/2-1]}\pi^{-1/4}\frac{-t^2-1/4}{2}\right]\left[e^{i\,\mathrm{Im}\ln\pi[s/2-1]}\pi^{-it/2}\zeta\left(\frac{1}{2}+it\right)\right]. \qquad (24)$$

Since the first term is always negative, the sign of the function is opposite to the sign of the second term, which is notated as

$$Z(t)=e^{i\vartheta(t)}\zeta\left(\frac{1}{2}+it\right) \qquad (25)$$

where $\vartheta(t)=\mathrm{Im}\ln\pi\left(\frac{it}{2}-\frac{3}{4}\right)-\frac{t}{2}\ln\pi$. Then, in order to determine the sign of $\xi(s)$, it is sufficient to evaluate $\vartheta(t)$ and $\zeta\left(\frac{1}{2}+it\right)$.

By simplification, $\vartheta(t)$ is computed as $\vartheta(t)=\frac{t}{2}\ln\frac{t}{2\pi}-\frac{t}{2}-\frac{\pi}{8}+\frac{1}{48t}+\frac{7}{5760t^3}+\dots$. Since the terms decrease very rapidly, the close approximation of $\vartheta(t)$ is

$$\vartheta(t)\sim\frac{t}{2}\ln\frac{t}{2\pi}-\frac{1}{2}-\frac{\pi}{8}+\frac{1}{48t}. \qquad (26)$$

[24] In fact, the algorithms in these methods were later used as a reference to build computer programs to generate more roots at a faster rate.

For $\zeta(s)$, applying Euler-Maclaurin summation to the series $\zeta(s) = \sum_{1}^{\infty} n^{-s}$, we get[25]

$$\zeta(s) \sim \sum_{n=1}^{N-1} n^{-s} + \frac{N^{1-s}}{s-1} + \frac{1}{2}N^{-s} + \frac{B_2}{2}sN^{-s-1} + \ldots + \frac{B_{2\nu}}{(2\nu)!}s(s+1)\ldots(s+2\nu-2)N^{-s-2\nu+1} \qquad (27)$$

where B_i is the Bernoulli polynomial that satisfies $B_n(x+1) - B_n(x) = n\,x^{n-1}$ (derived from $\int_x^{x+1} B_n(t)\,dt = x^n$) and N is a number that is large enough to make the terms of the series decrease rapidly in the magnitude.

The close approximations of $\vartheta(t)$ and $\zeta\left(\frac{1}{2}+it\right)$ in (26) and (27) determine the sign of $Z(t)$ and the sign of $\xi(t)$ also. Then, the interval where $\xi(t)$ changes signs can be narrowed down, which leads to comparatively good estimations of the non-trivial zeros of $\xi(t)$.

b. The Riemann-Siegel Formula:

Although it starts out with the same expression of $\xi\left(\frac{1}{2}+it\right)$ in (24) of the Euler-Maclaurin Summation method, the Riemann-Siegel Formula method put $Z(t)$ in a different form:

$$Z(t) = 2\sum_{n^2<\left(\frac{t}{2\pi}\right)} n^{-1/2}\cos\left[\vartheta(t) - t\ln n\right] + R \qquad (28)$$

where the remainder term $R \sim \frac{e^{-i\vartheta(t)}e^{-t\pi/2}}{(2\pi)^{1/2}(2\pi)^{it}e^{-i\pi/4}(1-ie^{-\pi t})}\int_{L_1}\frac{(-x)^{-1/2+it}e^{-Nx}dx}{e^x-1}$; N is the integer part of $(t/2\pi)^{1/2}$ and L_1 is a line segment in the complex x-plane which has slope 1, length $(2\pi t)^{1/2}$, and midpoint on the imaginary axis at $i(2\pi t)^{1/2}$.

The Riemann-Siegel formula is used to evaluate the numerical value of R, and thus, ultimately the value of $Z(t)$ and $\xi\left(\frac{1}{2}+it\right)$. To a first order approximation, carried out by the Riemann-Siegel formula, the remainder R is:

$$R \sim (-1)^{N-1}\left(\frac{t}{2\pi}\right)^{-1/4}\frac{\cos 2\pi\left(p^2 - p - \frac{1}{16}\right)}{\cos 2\pi p}$$ where p is the fractional part of $(t/2\pi)^{1/2}$. (29)

[25] Euler-Maclaurin summation is a technique for numerical evaluation of sums that involves the "Bernoulli numbers."

Higher order approximations, which return better approximations, can be obtained by using the higher order terms of the series expansion of the remainder R although it might get very complicated.

3. Sample Computations

a. Euler-Maclaurin Summation:

The box below shows the summary of finding the approximation of $\zeta\left(\frac{1}{2}+18i\right)$ using the Euler-Maclaurin Summation.

$$1^{-s} = 1.00000$$
$$2^{-s} = +0.70427 + 0.06336i$$
$$3^{-s} = +0.34726 - 0.46124i$$
$$4^{-s} = +0.49199 + 0.08924i$$
$$5^{-s} = -0.34333 - 0.28657i$$
$$6^{1-s}(s-1)^{-1} = -0.10340 - 0.08839i$$
$$\frac{1}{2}6^{-s} = +0.13689 - 0.15142i$$
$$B_2 = +0.07761 + 0.06634i$$
$$B_4 = +0.01372 + 0.00725i$$
$$B_6 = +0.00349 + 0.00017i$$
$$B_8 = +0.00072 - 0.00053i$$
$$\zeta\left(\frac{1}{2}+18i\right) \sim 2.32922 - 0.18865i$$

On the other hand, $\vartheta(18)$ is evaluated as:

$$\vartheta(18) \sim 9\ln\frac{9}{\pi} - 9 - \frac{\pi}{8} + \frac{1}{48.18} = 9.472452 - 9 - 0.392699 + 0.001158 = 0.080911$$

Thus, $Z(18) = e^{0.0809i}(2.329 - 0.189i) = 2.337 + 0.000i$, which is positive and implies that $\xi\left(\frac{1}{2}+18i\right)$ is negative. Because $\xi\left(\frac{1}{2}\right)$ is positive, we know that there exists one root ρ lying between ½ and ½ + 18i. Repeating the process for multiple times, we can find the first root of the zeta function, which is ρ = 14.1347.

b. The Riemann-Siegel Formula:

Based on the formula (28) and (29) of $Z(t)$ and R, we compute the first order approximation of Z(18) as follows.

$(t/2\pi)^{1/2} = 1.692569$, so $N = 1$, and $p = 0.692569$.

$$\text{Then } Z(18) \sim 2\cos\vartheta(18) + (-1)^{1-1}\left(\frac{18}{2\pi}\right)^{-1/4} \frac{\cos 2\pi\left(0.692569^2 - 0.692569 - \frac{1}{16}\right)}{\cos(2\pi 0.692569)}$$

$$= 1.993457 + (0.768647)\frac{-0.159022}{-0.353070} = 1.993457 + 0.346197 = 2.339654$$

This is the first approximation of $Z(18)$. Compared with the value obtained by Euler-Maclaurin Summation, the Riemann-Siegel Formula gives better accuracy.

Although these methods are tedious, the fact that they have been used to locate a large amount of the non-trivial zeros of $\xi(t)$ on the critical line contributes to the plausibility of the unsolved Riemann Hypothesis.

4. Attempted Proofs of the Riemann Hypothesis

The Riemann Hypothesis remains unproved even till now, and therefore it can only be considered as a hypothesis. Because of this fascinating challenge and the importance of the hypothesis, many mathematicians have been trying to prove it over 200 years. 1885 was the year of the first significant failed attempt at a proof. Stieltjes claimed to solve the hypothesis by using Mertens Conjecture, which states that the Mertens function M(n) is bounded by $\sqrt{n}$ and thus, implies the Riemann Hypothesis. Although this conjecture was supported by a large amount of numerical evidence, it was disproved by Odlyzko and te Riele in 1985. Therefore, the Riemann Hypothesis still stays as an interesting elusive problem that is waiting for a valid reply. Fortunately, the good news is that there have been a lot of computational supports while no counterexample has yet been found.

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