AN APPLICATION OF THE IMPLICIT FUNCTION THEOREM TO COMPARATIVE STATICS ANALYSIS

ABSTRACT. Comparative statics analysis is concerned with the comparison of equilibriums that are associated with different sets of values of exogenous variables (parameters). The technique is crucial in economic analysis because most testable predictions and policy implications of economic models are generated by comparative static analysis. This research will investigate the Implicit Function Theorem and apply it to examine a comparative statics analysis of several economic models including the IS-LM model, the single-commodity market model with tax parameter, and the market model of two substitute goods.

1. MOTIVATION

A comparative statics analysis is sometimes quite easy to solve. For example, suppose the demand function of a linear, one-commodity market model is

$$Q_d = a - bP$$

and the supply function is

$$Q_s = c + dP$$

where P is price and a, b, c, d are positive independent parameters. The equilibrium price (P^*) and quantity (Q^*) are given when $Q_d = Q_s$. So, solving the equation $Q_s - Q_d = c + dP - (a - bP) = 0$, we get

$$\begin{aligned} P^* &= \frac{a-c}{b+d} \\ Q^* &= \frac{ad+bc}{b+d}. \end{aligned}$$

When we change the values of some of the parameters, we get a new equilibrium. Comparative statics analysis gives us an answer for the question: How would the new equilibrium compare with the old? If we can explicitly compute P^* and Q^* in terms of parameters (a, b, c, d) like the example above, then the answer is trivial. By taking partial derivatives of P^* and Q^* with respect to $a, b, c,$ and d, we can find the effect of change in each parameter on our equilibrium. For example, $\frac{\partial P^*}{\partial a} = \frac{1}{b+d} > 0$, which means P^* and a are positively related, so that if we increase the parameter a while holding the others fixed, our equilibrium price P^* will increase.

However, this procedure is not always available since economic models are usually formulated using general functional forms. Even when we can express our equilibrium solution explicitly in terms of exogenous variables, computing its derivative is sometimes infeasible. In the following sections, we will employ several mathematical tools that will still enable us to calculate that type of derivative.

2. THE INVERSE FUNCTION THEOREM

The inverse function theorem plays a crucial role in our proof of the implicit function theorem because it states a conditions (continuity and non-singularity) for the existence of the inverse function at some points in the domain.

Theorem. *[p. 63, 1] Let A be open in $\mathbb{R}^n$; let $f : A \to \mathbb{R}^n$ be of class C^r, meaning the first r derivatives of f exist and are continuous. If $Df(\overrightarrow{x})$ is non-singular at some point $\overrightarrow{a}$ in A, then there is a neighborhood U of the point $\overrightarrow{a}$ such that f carries U in a one-to-one fashion onto an open set V of $\mathbb{R}^n$ and the inverse function $f^{-1} : V \to U$ is of class C^r.*

We should notice that the non-singularity of Df on A implies that f is locally one-to-one at each point of A, it does not imply that f is (globally) one-to-one on all of A.

3. Implicit function

Definition. An equation of the form

$$f(x, y) = 0$$

implicitly defines y as a function of x on a domain A if there is a function g on A for which $f(x, g(x)) = 0$, for all $x \in A$.

Notice that the use of zero in the above equation serves to simplify notation, but is not essential. The condition $f(x, y) = 0$ is equivalent to $h(x, y) = p$ where $h(x, y) = f(x, y) + p$.

A classic example comes from the graph of a circle, For instance,

$$x^2 + y^2 - 4 = 0.$$

Now, we cannot describe y explicitly in terms of x, because for each x in $[-2, 2]$, we have two choices of y such that $y = \sqrt{(4 - x^2)}$ and $y = -\sqrt{(4 - x^2)}$. However, we know that around some fixed points in [-2,2], say $(\sqrt{3}, 1)$, we can describe $y = \sqrt{(4 - x^2)}$ and estimate the effect of a small change in x on y. This process is exactly what we need in the comparative statics analysis, in which we have a model that is in equilibrium (a fixed point) and we want to know what happens to our equilibrium if we change parameters of the model.

The problem here is that at some points, we cannot define our endogenous variable in terms of the exogenous variables. For the above example, if we choose point (2, 0) and decrease x by a small value, we do not know whether to choose $y = \sqrt{(4 - x^2)}$ or $y = -\sqrt{(4 - x^2)}$, and therefore, we cannot estimate the effect of the change in x on y.

Hence, to solve the comparative statics analysis of more complicated economic models, we want to know under what conditions, we can estimate the effect of a small change in the exogenous variables on our endogenous variables. In mathematical language, we want to know for a given equation of the form $f(x_1, \ldots, x_n, y) = 0$, when we can take the partial derivatives of y with respect to $x_1, \ldots, x_n$. Surprisingly, that condition is quite simple. We will investigate this in the next section by proving the implicit function theorem.

4. The implicit function theorem

The following theorem is the most general version of the Implicit Function Theorem of real-valued functions. We will deal with a function of multiple variables, from $\mathbb{R}^{k+n} \to \mathbb{R}^n$, instead of a single-variable function. The theorem and the proof are excerpted (with modification) from Munkres's *Analysis on Manifolds*.

Theorem. *[p. 71, 1] Let A be open in $\mathbb{R}^{k+n}$ and let $f : A \to \mathbb{R}^n$ be of class C^r. Write f in the form $f(\overrightarrow{x}, \overrightarrow{y})$, for $\overrightarrow{x} = (x_1, ..., x_k) \in \mathbb{R}^k$ and $\overrightarrow{y} = (y_1, ..., y_n) \in \mathbb{R}^n$. Suppose that $(\overrightarrow{a}, \overrightarrow{b})$ is a point of A such that $f(\overrightarrow{a}, \overrightarrow{b}) = \overrightarrow{0}$ and*

$$det \frac{\partial f}{\partial \overrightarrow{y}}(\overrightarrow{a}, \overrightarrow{b}) \neq \overrightarrow{0}.$$

Then there is a neighborhood B of $\overrightarrow{a}$ in $\mathbb{R}^k$ and a unique continuous function $g : B \to \mathbb{R}^n$ such that $g(\overrightarrow{a}) = \overrightarrow{b}$ and

$$f(\overrightarrow{x}, g(\overrightarrow{x})) = \overrightarrow{0}$$

for all $\overrightarrow{x} \in B$. The function g is in fact of class C^r.

Moreover,

$$Dg(\overrightarrow{x}) = -\left[\frac{\partial f}{\partial \overrightarrow{y}}(\overrightarrow{x}, g(\overrightarrow{x}))\right]^{-1} \cdot \frac{\partial f}{\partial \overrightarrow{x}}(\overrightarrow{x}, g(\overrightarrow{x})).$$

Proof. Define $F : A \to \mathbb{R}^{k+n}$ by the equation

$$F(\overrightarrow{x}, \overrightarrow{y}) = (\overrightarrow{x}, f(\overrightarrow{x}, \overrightarrow{y})).$$

Notice that $\overrightarrow{x} = (x_1, ..., x_k) \in \mathbb{R}^k$, $\overrightarrow{y} = (y_1, ..., y_n) \in \mathbb{R}^n$ and $f : A \to \mathbb{R}^n$, so that we can write $f(\overrightarrow{x}, \overrightarrow{y}) = (f_1, ..., f_n) \in \mathbb{R}^n$and $F(\overrightarrow{x}, \overrightarrow{y}) = (F_1, F_2, ..., F_k, F_{k+1}, ..., F_{k+n}) = (x_1, ..., x_k, f_1, ..., f_n)$.

Then, let J be the Jacobian matrix of function F,

$$\begin{aligned} J = DF(\overrightarrow{x}, \overrightarrow{y}) &= DF(x_1, ..., x_k, y_1, ..., y_n) \\ &= \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial y_n} \\ \vdots & & \\ \frac{\partial F_{n+k}}{\partial x_1} & \cdots & \frac{\partial F_{n+k}}{\partial y_n} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial x_1}{\partial x_1} & \cdots & \frac{\partial x_1}{\partial x_k} & \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & (I_k) & \vdots & \vdots & (\overrightarrow{0}) & \vdots \\ \frac{\partial x_k}{\partial x_1} & \cdots & \frac{\partial x_k}{\partial x_k} & \frac{\partial x_k}{\partial y_1} & \cdots & \frac{\partial x_k}{\partial y_n} \\ \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_k} & \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \vdots & (\frac{\partial f}{\partial \overrightarrow{x}}) & \vdots & \vdots & (\frac{\partial f}{\partial \overrightarrow{y}}) & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_k} & \frac{\partial f_n}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_n} \end{bmatrix} \\ &= \begin{bmatrix} I_k & \overrightarrow{0} \\ \frac{\partial f}{\partial \overrightarrow{x}} & \frac{\partial f}{\partial \overrightarrow{y}} \end{bmatrix}. \end{aligned}$$

Consider the determinant evaluating formula:

$$detM = \sum_{k=1}^{n} (-1)^{i+k} m_{ik} \cdot detM_{ik}$$

where M is an n by n matrix, i is fixed, and M_{ik} is the (i, k)-minor of M.

Look at the first row of J, since only the first entry is non zero (in fact, $j_{11} = 1$), we get $detJ = (-1)^2 j_{11} detJ_{11} = detJ_{11}$. Similarly, by repeatedly applying the above formula k

times to J and using $j_{11} = j_{22} = j_{33} = ... = j_{kk} = 1$, we get

$$\begin{aligned} detDF(\overrightarrow{x}, \overrightarrow{y}) &= detJ \\ &= detJ_{11} \\ &= detJ_{22} \\ &= ... \\ &= detJ_{kk} \\ &= det\frac{\partial f}{\partial \overrightarrow{y}} \end{aligned}$$

So that the determinant of Jacobian matrix of function F is non-singular at the point $(\overrightarrow{a}, \overrightarrow{b})$ as we assumed $det\frac{\partial f}{\partial \overrightarrow{y}}(\overrightarrow{a}, \overrightarrow{b}) \neq 0$. Applying the inverse function theorem to the map F, there exists a neighborhood U of $\overrightarrow{a} \in \mathbb{R}^n$ and V of $\overrightarrow{b} \in \mathbb{R}^k$ such that F carries $U \times V$ in a one-to-one fashion onto an open set W in $\mathbb{R}^{k+n}$ and the inverse function $G : W \to U \times V$ is of class C^r.

Let $t : W \to \mathbb{R}^k$ and $h : W \to \mathbb{R}^n$ denote the component functions of G, such that $G(\overrightarrow{s}, \overrightarrow{z}) = (t(\overrightarrow{s}, \overrightarrow{z}), h(\overrightarrow{s}, \overrightarrow{z}))$ for all $(\overrightarrow{s}, \overrightarrow{z}) \in W \subset \mathbb{R}^{k+n}$. Then

$$\begin{aligned} (\overrightarrow{s}, \overrightarrow{z}) &= F(G(\overrightarrow{s}, \overrightarrow{z})) \\ &= F(t(\overrightarrow{s}, \overrightarrow{z}), h(\overrightarrow{s}, \overrightarrow{z})) \\ &= (t(\overrightarrow{s}, \overrightarrow{z}), f(t(\overrightarrow{s}, \overrightarrow{z}), h(\overrightarrow{s}, \overrightarrow{z})), \end{aligned}$$

which shows that $\overrightarrow{s} = t(\overrightarrow{s}, \overrightarrow{z})$ and $\overrightarrow{z} = f(t(\overrightarrow{s}, \overrightarrow{z}), h(\overrightarrow{s}, \overrightarrow{z})) = f(\overrightarrow{s}, h(\overrightarrow{s}, \overrightarrow{z}))$. Thus, for all $(\overrightarrow{s}, \overrightarrow{z}) \in W$, we can write

$$(\overrightarrow{s}, \overrightarrow{z}) = (\overrightarrow{s}, f(\overrightarrow{s}, h(\overrightarrow{s}, \overrightarrow{z}))) \tag{4.1}$$

and

$$G(\overrightarrow{s}, \overrightarrow{z}) = (\overrightarrow{s}, h(\overrightarrow{s}, \overrightarrow{z})) \tag{4.2}$$

Since $f(\overrightarrow{a}, \overrightarrow{b}) = \overrightarrow{0}$, we get $F(\overrightarrow{a}, \overrightarrow{b}) = (\overrightarrow{a}, f(\overrightarrow{a}, \overrightarrow{b})) = (\overrightarrow{a}, \overrightarrow{0})$. Thus, W contains $(\overrightarrow{a}, \overrightarrow{0})$. Let B be a connected neighborhood of $\overrightarrow{a} \in \mathbb{R}^k$, chosen small enough that $B \times \overrightarrow{0}$ is contained in W.

Let $g : B \to \mathbb{R}^n$ be given by $g(\overrightarrow{x}) = h(\overrightarrow{x}, \overrightarrow{0})$. If $\overrightarrow{x} \in B$, then $(\overrightarrow{x}, \overrightarrow{0}) \in W$.

Using the equation 4.1 and letting $\overrightarrow{z} = \overrightarrow{0}$, we have:

$$(\overrightarrow{x}, \overrightarrow{0}) = (\overrightarrow{x}, f(\overrightarrow{x}, h(\overrightarrow{x}, \overrightarrow{0})).$$

Thus,

$$\begin{aligned} \overrightarrow{0} &= f(\overrightarrow{x}, h(\overrightarrow{x}, \overrightarrow{0})) \\ &= f(\overrightarrow{x}, g(\overrightarrow{x})). \end{aligned}$$

Then g satisfies the equation $f(\overrightarrow{x}, g(\overrightarrow{x})) = \overrightarrow{0}$, as desired.

Moreover, since $F(\overrightarrow{a}, \overrightarrow{b}) = (\overrightarrow{a}, \overrightarrow{0})$, applying the inverse function G to $(\overrightarrow{a}, \overrightarrow{0})$, we get

$$\begin{aligned} (\overrightarrow{a}, \overrightarrow{b}) &= G(\overrightarrow{a}, \overrightarrow{0}) \\ &= (\overrightarrow{a}, h(\overrightarrow{a}, \overrightarrow{0})) \text{ (by 4.2)} \\ &= (\overrightarrow{a}, g(\overrightarrow{a})) \end{aligned}$$

which shows $g(\overrightarrow{a}) = \overrightarrow{b}$, as desired.

Thus, there exists a continuous function $g : B \to \mathbb{R}^n$ such that $g(\overrightarrow{a}) = \overrightarrow{b}$ and

$$f(\overrightarrow{x}, g(\overrightarrow{x})) = \overrightarrow{0} \text{ for all } \overrightarrow{x} \in \text{B}.$$

To prove the uniqueness of the function g, let $g_0 : B \to \mathbb{R}^n$ be a continuous function such that $g_0(\overrightarrow{a}) = \overrightarrow{b}$ and $f(\overrightarrow{x}, g_0(\overrightarrow{x})) = \overrightarrow{0}$ for all $\overrightarrow{x} \in B$.

Let

$$A_1 = \{\overrightarrow{x} \in B : |g(\overrightarrow{x}) - g_0(\overrightarrow{x})| > 0\}$$

and

$$A_2 = \{\overrightarrow{x} \in B : |g(\overrightarrow{x}) - g_0(\overrightarrow{x})| = 0\}.$$

We want to prove A_1 and A_2 are both open in B.

Since g , g_0 and the norm are continuous, it follows that $|g(\overrightarrow{x}) - g_0(\overrightarrow{x})|$ is a continuous composite function mapping B to $\mathbb{R}$. Hence, given $(0, \infty)$ is open in $\mathbb{R}$, the set $A_1 = \{\overrightarrow{x} \in B : |g(\overrightarrow{x}) - g_0(\overrightarrow{x})| > 0\}$ is open in B.

Let $\overrightarrow{a_0}$ be an arbitrary point in A_2, which implies $g_0(\overrightarrow{a_0}) = g(\overrightarrow{a_0}) = h(\overrightarrow{a_0}, \overrightarrow{0}) \in V$ (since h is the component function of G). Since V is open, g_0 is continuous and $g_0(\overrightarrow{a_0}) \in V$, there is a neighborhood B_0 of $\overrightarrow{a_0}$ contained in B such that g_0 maps B_0 into V. For all $\overrightarrow{x} \in B_0 \subset B$, since $f(\overrightarrow{x}, g_0(\overrightarrow{x})) = \overrightarrow{0}$, we get $F(\overrightarrow{x}, g_0(\overrightarrow{x})) = (\overrightarrow{x}, f(\overrightarrow{x}, g_0(\overrightarrow{x})) = (\overrightarrow{x}, \overrightarrow{0})$. Since $(\overrightarrow{x}, 0) \in W$, applying the function $G : W \to U \times V$ to $(\overrightarrow{x}, \overrightarrow{0})$, and using the fact that G and F are inverse functions of each other, we get:

$$\begin{aligned} (\overrightarrow{x}, g_0(\overrightarrow{x})) &= G(\overrightarrow{x}, \overrightarrow{0}) \\ &= (\overrightarrow{x}, h(\overrightarrow{x}, \overrightarrow{0})) \text{ (by 4.2)}, \end{aligned}$$

so $g_0(\overrightarrow{x}) = h(\overrightarrow{x}, \overrightarrow{0}) = g(\overrightarrow{x})$ for $\overrightarrow{x} \in B_0$.

Thus, for any point in A_2, there exists an open subset of B containing that point and hence, A_2 is open in B.

Since $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = B$, the fact B that is connected and A_2 is non-empty ($\overrightarrow{a} \in A_2$) imply A_1 is empty.

Hence, g_0 agrees with g in B and g is unique.

To find the derivative of the function g, define $p : B \to \mathbb{R}^{k+n}$ such that $p(\overrightarrow{x}) = (\overrightarrow{x}, g(\overrightarrow{x}))$ and $P : B \to \mathbb{R}^n$ such that $P(\overrightarrow{x}) = f(p(\overrightarrow{x}))$, so that $P(\overrightarrow{x}) = f(\overrightarrow{x}, g(\overrightarrow{x}))$.

For all $\overrightarrow{x} \in B$, since $f(\overrightarrow{x}, g(\overrightarrow{x})) = \overrightarrow{0}$, it follows that $P(\overrightarrow{x}) = \overrightarrow{0}$. Using the chain rule, differentiate both side of the equation $P(\overrightarrow{x}) = \overrightarrow{0}$, we get

$$\begin{aligned} \overrightarrow{0} &= DP(\overrightarrow{x}) \\ &= Df(p(\overrightarrow{x})) \cdot Dp(\overrightarrow{x}). \end{aligned} \tag{4.3}$$

Recall that the function f maps $A \in \mathbb{R}^{k+n}$ to $\mathbb{R}^n$ and we write f in the form $f(\overrightarrow{x}, \overrightarrow{y})$ for $\overrightarrow{x} \in \mathbb{R}^k$ and $\overrightarrow{y} \in \mathbb{R}^n$, so

$$Df(p(\overrightarrow{x})) = \left[\begin{array}{cc} \dfrac{\partial f}{\partial \overrightarrow{x}}(p(\overrightarrow{x})) & \dfrac{\partial f}{\partial \overrightarrow{y}}(p(\overrightarrow{x})) \end{array} \right].$$

Also recall that the function g maps $B \in \mathbb{R}^k$ to $\mathbb{R}^n$, so we can rewrite $g(\overrightarrow{x}) = [g_1(\overrightarrow{x}), ..., g_n(\overrightarrow{x})]$ and $p(\overrightarrow{x}) = [x_1, ..., x_k, g_1(\overrightarrow{x}), ..., g_n(\overrightarrow{x})]$. Thus, the derivative of p with respect to $\overrightarrow{x} =$

$(x_1, ..., x_k) \in \mathbb{R}^k$ is

$$
\begin{aligned}
Dp(\overrightarrow{x}) &= \begin{bmatrix} \frac{\partial x_1}{\partial x_1} & \cdots & \frac{\partial x_1}{\partial x_k} \\ \vdots & (I_k) & \\ \frac{\partial x_k}{\partial x_1} & \cdots & \frac{\partial x_k}{\partial x_k} \\ \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_k} \\ \vdots & (Dg(\overrightarrow{x})) & \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_k} \end{bmatrix} \\
&= \begin{bmatrix} I_k \\ Dg(\overrightarrow{x}) \end{bmatrix}.
\end{aligned}
$$

Substitute $Df(p(\overrightarrow{x})$ and $Dp(\overrightarrow{x})$ to the equation 4.3, we get

$$
\begin{aligned}
0 &= \begin{bmatrix} \frac{\partial f}{\partial \overrightarrow{x}}(p(\overrightarrow{x})) & \frac{\partial f}{\partial \overrightarrow{y}}(p(\overrightarrow{x})) \end{bmatrix} . \begin{bmatrix} I_k \\ Dg(\overrightarrow{x}) \end{bmatrix} \\
&= \frac{\partial f}{\partial \overrightarrow{x}}(p(\overrightarrow{x})) + \frac{\partial f}{\partial \overrightarrow{y}}(p(\overrightarrow{x})).Dg(\overrightarrow{x}).
\end{aligned}
$$

So,

$$
Dg(\overrightarrow{x}) = -\left[\frac{\partial f}{\partial \overrightarrow{y}}(\overrightarrow{x}, g(\overrightarrow{x}))\right]^{-1} . \frac{\partial f}{\partial \overrightarrow{x}}(\overrightarrow{x}, g(\overrightarrow{x})).
$$

□

The Implicit Function Theorem is extremely useful in economics to have a quick analysis of the effect of changes in exogenous on endogenous variables without the need of explicit derivation. In the next following sections, we will apply the theorem to analyze some economic models such as the IS - LM, the market model of one good with tax parameter, and the market model of two substitute goods.

5. The Investment Saving - Liquidity Preference Money Supply model

In this section, we will use the Implicit Function Theorem to carry out a comparative statics analysis of the IS-LM equilibrium. For simplicity, we will assume the economy is closed (meaning there are no export and import factors), and contains only two markets: the market for goods and services (IS), and the market for money (LM).

The IS curve describes equilibrium in the goods market

$$y = C(y, r) + I(r) + G$$

with national income y equal to aggregate output, consisting of consumption C, investment I, and government spending G. Consumption $C(y, r)$ depends on income y and the real interest rate r. We assume that consumption would increase if national income increased and the real interest rate decreased (i.e. $\frac{\partial C}{\partial y} > 0$ and $\frac{\partial C}{\partial r} < 0$). Moreover, we assume that $\frac{\partial C}{\partial y} < 1$ as a consumer would spend less than one dollar, given he/she gets one dollar increase in income. We also assume that investment would increase if the real interest rate decreased (i.e. $\frac{\partial I}{\partial r} > 0$).

The LM curve describes equilibrium in the money market

$$M = L(y, r)$$

when the real money supply M is equal to the real demand for money, which depends on national income y and the real interest rate r. We assume that the real demand for money would increase if national income increased and the real interest rate decreased (i.e. $\frac{\partial L}{\partial y} > 0$ and $\frac{\partial L}{\partial r} < 0$).

A short summary of our model assumptions is followed

$$
\begin{aligned}
1 > \frac{\partial C}{\partial y} &> 0 \\
\frac{\partial C}{\partial r} &< 0 \\
\frac{\partial I}{\partial r} &> 0 \\
\frac{\partial L}{\partial y} &> 0 \\
\frac{\partial L}{\partial r} &< 0.
\end{aligned}
\tag{5.1}
$$

Analysis of this model consists of examining the impact of changes in the exogenous variables G, M on the dependent variables r, y. Designating two functions of equilibrium, we can analyze the IS - LM model as a function f from $\mathbb{R}^4$ to $\mathbb{R}^2$, given by $f = (f_1, f_2) = (0, 0)$ when two markets reach equilibrium.

$$
\begin{aligned}
f_1(G, M, r, y) &= y - C(y, r) - I(r) - G = 0 \\
f_2(G, M, r, y) &= L(y, r) - M = 0.
\end{aligned}
$$

Let J be the derivative of f with respect to $\overrightarrow{y} = (r, y)$

$$
J = \frac{\partial f}{\partial \overrightarrow{y}} = \begin{bmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} -\left(\frac{\partial C}{\partial r} + \frac{\partial I}{\partial r}\right) & \left(1 - \frac{\partial C}{\partial y}\right) \\ \left(\frac{\partial L}{\partial r}\right) & \left(\frac{\partial L}{\partial y}\right) \end{bmatrix}.
$$

Based on our assumptions 5.1, since $\frac{\partial C}{\partial r}, \frac{\partial I}{\partial r} < 0$, $\frac{\partial C}{\partial y} < 1$, $\frac{\partial L}{\partial r} < 0$, and $\frac{\partial L}{\partial y} > 0$, we get

$$
detJ = -\left(\frac{\partial C}{\partial r} + \frac{\partial I}{\partial r}\right)_{<0} \left(\frac{\partial L}{\partial y}\right)_{>0} - \left(1 - \frac{\partial C}{\partial y}\right)_{>0} \left(\frac{\partial L}{\partial r}\right)_{<0} > 0.
$$

Let $\overrightarrow{x} = (G, M) \in U \subset \mathbb{R}^2$ and $\overrightarrow{y} = (r, y) \in V \subset \mathbb{R}^2$, then the function $f : U \times V \to \mathbb{R}^2$ satisfies the condition of the Implicit Function Theorem $det \frac{\partial f}{\partial \overrightarrow{y}} \neq 0$. Thus, in a neighborhood B around any equilibrium, there exists a function $g : B \to \mathbb{R}^2$ specifying r and y as a function of G and M. Although the function g cannot be explicitly written in terms of G and M, we can derive the derivative of g with respect to $\overrightarrow{x} = (G, M)$ by the formula

$$
Dg(\overrightarrow{x}) = -\left[\frac{\partial f}{\partial \overrightarrow{y}}(\overrightarrow{x}, g(\overrightarrow{x}))\right]^{-1} \cdot \frac{\partial f}{\partial \overrightarrow{x}}(\overrightarrow{x}, g(\overrightarrow{x})).
$$

So,

$$
\begin{aligned}
Dg(\vec{x}) &= -\begin{bmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial y} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial f_1}{\partial G} & \frac{\partial f_1}{\partial M} \\ \frac{\partial f_2}{\partial G} & \frac{\partial f_2}{\partial M} \end{bmatrix} \\
&= -\frac{1}{detJ} \begin{bmatrix} \left(\frac{\partial L}{\partial y}\right) & -\left(1 - \frac{\partial C}{\partial y}\right) \\ -\left(\frac{\partial L}{\partial r}\right) & -\left(\frac{\partial C}{\partial r} + \frac{\partial I}{\partial r}\right) \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \frac{1}{detJ} \begin{bmatrix} \left(\frac{\partial L}{\partial y}\right) & -\left(1 - \frac{\partial C}{\partial y}\right) \\ -\left(\frac{\partial L}{\partial r}\right) & -\left(\frac{\partial C}{\partial r} + \frac{\partial I}{\partial r}\right) \end{bmatrix}.
\end{aligned}
$$

Since g specifies r and y as a function of G and M, we get $Dg(\vec{x}) = \begin{bmatrix} \frac{\partial r}{\partial G} & \frac{\partial r}{\partial M} \\ \frac{\partial y}{\partial G} & \frac{\partial y}{\partial M} \end{bmatrix}$,

and so

$$
\begin{bmatrix} \frac{\partial r}{\partial G} & \frac{\partial r}{\partial M} \\ \frac{\partial y}{\partial G} & \frac{\partial y}{\partial M} \end{bmatrix} = \frac{1}{detJ} \begin{bmatrix} \left(\frac{\partial L}{\partial y}\right) & -\left(1 - \frac{\partial C}{\partial y}\right) \\ -\left(\frac{\partial L}{\partial r}\right) & -\left(\frac{\partial C}{\partial r} + \frac{\partial I}{\partial r}\right) \end{bmatrix}.
$$

Hence, we can draw several conclusions of the IS-LM model about the impact of G and M on r and Y:

1. $\frac{\partial r}{\partial G} = \frac{\partial L/\partial y}{detJ} > 0$ implies that an increase in government spending would yield an increase in the real interest rate.

2. $\frac{\partial y}{\partial G} = \frac{-(\partial L/\partial r)}{detJ} > 0$ implies that an increase in government spending would yield an increase in national income.

3. $\frac{\partial r}{\partial M} = \frac{-(1 - \partial C/\partial y)}{detJ} < 0$ implies that an increase in the real money supply would yield an decrease in the real interest rate.

4. $\frac{\partial y}{\partial M} = \frac{-(\partial C/\partial r + \partial I/\partial r)}{detJ} > 0$ implies that an increase in the real money supply would yield an increase in national income.

6. A SIMPLE SET UP OF THE IS-LM MODEL

In section 5, our consumption and investment functions are in the general form, so that it is not possible to derive r and y in terms of G and M. This section will examine a simple case of the IS-LM model, which we can explicitly write r and y as a function of G and M. By doing so, we will be able to compare the result derived by both "explicit" and "implicit" approach.

For simplicity, we assume

$$
\begin{aligned}
C(y, r) &= a_C y + b_C r + c_C \\
I(r) &= b_I r + c_I \\
L(y, r) &= a_L Y + b_L r + c_L
\end{aligned}
$$

where $a_C, a_L > 0, b_C, b_I, b_L < 0$, and $a_C < 1$ by the construction of our model described in the previous section.

Then, the function $f = (f_1, f_2)$ from $\mathbb{R}^4$ to $\mathbb{R}^2$ describing equilibrium in the goods and money markets is written as

$$\begin{aligned} f_1(G, M, r, y) &= y - C(y, r) - I(r) - G \\ &= y - (a_C y + b_C r + c_C) - (b_I r + c_I) - G \\ &= (1 - a_C)y - (b_C + b_I)r - (c_C + c_I) - G \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} f_2(G, M, r, y) &= L(y, r) - M \\ &= a_L y + b_L r + c_L - M \\ &= 0. \end{aligned}$$

So,

$$G = (1 - a_C)y - (b_C + b_I)r - (c_C + c_I) \tag{6.1}$$
$$M = a_L y + b_L r + c_L. \tag{6.2}$$

To derive r as a function of G and M, we multiply both side of the equation 6.1 with a_L and the equation 6.2 with $(1 - a_C)$

$$\begin{aligned} Ga_L &= (1 - a_C)a_L y - (b_C + b_I)a_L r - (c_C + c_I)a_L \\ M(1 - a_C) &= (1 - a_C)a_L y + (1 - a_C)b_L r + c_L(1 - a_C) \end{aligned}$$

then,

$$\begin{aligned} Ga_L - M(1 - a_C) &= (1 - a_C)a_L y - (b_C + b_I)a_L r - (c_C + c_I)a_L \\ &\quad - (1 - a_C)a_L y - (1 - a_C)b_L r - c_L(1 - a_C) \\ &= -r[(b_C + b_I)a_L + (1 - a_C)b_L] - [(c_C + c_I)a_L + c_L(1 - a_C)]. \end{aligned}$$

So, we can directly derive r as a function of G and M

$$r = \frac{-a_L}{(b_C + b_I)a_L + (1 - a_C)b_L} G + \frac{1 - a_C}{(b_C + b_I)a_L + (1 - a_C)b_L} M - \frac{(c_C + c_I)a_L + c_L(1 - a_C)}{(b_C + b_I)a_L + (1 - a_C)b_L}.$$

Then, the derivative of r with respect to G and M can be calculated as

$$\begin{aligned} \frac{\partial r}{\partial G} &= \frac{-a_L}{(b_C + b_I)_{(<0)} a_{L(>0)} + (1 - a_C)_{(>0)} b_{L(<0)}} > 0 \text{ and} \\ \frac{\partial r}{\partial M} &= \frac{1 - a_C}{(b_C + b_I)a_L + (1 - a_C)b_L} < 0. \end{aligned}$$

Similarly, to derive y in terms of G and M, we multiply both side of the equation (6.1) with b_L and the equation (6.2) with $(b_C + b_I)$

$$\begin{aligned} Gb_L &= (1 - a_C)b_L y - (b_C + b_I)b_L r - (c_C + c_I)b_L \\ M(b_C + b_I) &= a_L(b_C + b_I)y + b_L(b_C + b_I)r + c_L(b_C + b_I) \end{aligned}$$

then,

$$\begin{aligned} Gb_L + M(b_C + b_I) &= (1 - a_C)b_L y - (b_C + b_I)b_L r - (c_C + c_I)b_L \\ &\quad + a_L(b_C + b_I)y + b_L(b_C + b_I)r + c_L(b_C + b_I) \\ &= y[(1 - a_C)b_L + a_L(b_C + b_I)] - [(c_C + c_I)b_L - c_L(b_C + b_I)]. \end{aligned}$$

So, we can explicitly solve for y as a function of G and M

$$y = \frac{b_L}{(1-a_C)b_L + a_L(b_C+b_I)}G + \frac{(b_C+b_I)}{(1-a_C)b_L + a_L(b_C+b_I)}M + \frac{(c_C+c_I)b_L - c_L(b_C+b_I)}{(1-a_C)b_L + a_L(b_C+b_I)}.$$

Thus, the derivative of y with respect to G and M is

$$\begin{aligned}\frac{\partial y}{\partial G} &= \frac{b_L}{(1-a_C)b_L + a_L(b_C+b_I)} > 0\\ \frac{\partial y}{\partial M} &= \frac{(b_C+b_I)}{(1-a_C)b_L + a_L(b_C+b_I)} > 0.\end{aligned}$$

Now, we want to compare this "explicit" approach with the "implicit" method. The function $f : \mathbb{R}^4 \to \mathbb{R}^2$ is stated as

$$\begin{aligned} f_1(G, M, r, y) &= (1-a_C)y - (b_C+b_I)r - (c_C+c_I) - G = 0\\ f_2(G, M, r, y) &= a_L y + b_L r + c_L - M = 0.\end{aligned}$$

Let $\overrightarrow{x} = (G, M) \in \mathbb{R}^2$ and $\overrightarrow{y} = (r, y) \in \mathbb{R}^2$, then

$$\begin{aligned}\frac{\partial f}{\partial \overrightarrow{y}} &= \begin{bmatrix} \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial y}\end{bmatrix}\\ &= \begin{bmatrix} -(b_C+b_I) & (1-a_C) \\ b_L & a_L\end{bmatrix}.\end{aligned}$$

So, $det\frac{\partial f}{\partial \overrightarrow{y}} = -a_L(b_C+b_I) - (1-a_C)b_L > 0$. Thus, by the Implicit function theorem, there exists a function g specifying r and y as a function of G and M and the derivative of the function g can be formulated as

$$Dg(\overrightarrow{x}) = -\left[\frac{\partial f}{\partial \overrightarrow{y}}(\overrightarrow{x}, g(\overrightarrow{x}))\right]^{-1} \cdot \frac{\partial f}{\partial \overrightarrow{x}}(\overrightarrow{x}, g(\overrightarrow{x})).$$

Since $\frac{\partial f}{\partial \overrightarrow{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial G} & \frac{\partial f_1}{\partial M} \\ \frac{\partial f_2}{\partial G} & \frac{\partial f_2}{\partial M}\end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1\end{bmatrix}$ and $\frac{\partial f}{\partial \overrightarrow{y}} = \begin{bmatrix} -(b_C+b_I) & (1-a_C) \\ b_L & a_L\end{bmatrix}$, we get

$$\begin{aligned} Dg(\overrightarrow{x}) &= -\begin{bmatrix} -(b_C+b_I) & (1-a_C) \\ b_L & a_L\end{bmatrix}^{-1}\begin{bmatrix} -1 & 0 \\ 0 & -1\end{bmatrix}\\ &= \frac{1}{a_L(b_C+b_I)+(1-a_C)b_L}\begin{bmatrix} a_L & -(1-a_C) \\ -b_L & -(b_C+b_I)\end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1\end{bmatrix}\\ &= \frac{1}{a_L(b_C+b_I)+(1-a_C)b_L}\begin{bmatrix} -a_L & (1-a_C) \\ b_L & (b_C+b_I)\end{bmatrix}.\end{aligned}$$

Thus, $Dg(\overrightarrow{x}) = \begin{bmatrix} \frac{\partial r}{\partial G} & \frac{\partial r}{\partial M} \\ \frac{\partial y}{\partial G} & \frac{\partial y}{\partial M} \end{bmatrix} = \frac{1}{a_L(b_C + b_I) + (1 - a_C)b_L} \begin{bmatrix} -a_L & (1 - a_C) \\ b_L & (b_C + b_I) \end{bmatrix}$, which means

$$\begin{aligned} \frac{\partial r}{\partial G} &= \frac{-a_L}{a_L(b_C + b_I) + (1 - a_C)b_L} > 0 \\ \frac{\partial r}{\partial M} &= \frac{(1 - a_C)}{a_L(b_C + b_I) + (1 - a_C)b_L} < 0 \\ \frac{\partial y}{\partial G} &= \frac{b_L}{a_L(b_C + b_I) + (1 - a_C)b_L} > 0 \\ \frac{\partial y}{\partial M} &= \frac{(b_C + b_I)}{a_L(b_C + b_I) + (1 - a_C)b_L} > 0. \end{aligned}$$

Indeed, the result derived by the Implicit Function Theorem coincides with the "explicit" result.

7. A MARKET MODEL WITH TAX PARAMETER

In section 1, we have seen that a simple linear market model can be easily solved explicitly. Now, we would like to relax the linearity assumption and introduce a new exogenous variable, the tax rate (t). Then, by using the Implicit Function Theorem, we will be able to find the effect of a change in the tax rate on our price equilibrium.

Since consumers will reduce their purchase when the price increases, we suppose that quantity demanded q_d for some commodity is inversely related to the price p according the demand function d

$$q_d = d(p)$$

where $\frac{\partial d}{\partial p} < 0$.

Also, we assume that quantity supplied q_s is positively related to the price p and negatively related to the tax rate t since suppliers will be willing to produce more when the price increases and the tax rate decreases. Therefore,

$$q_s = s(p, t)$$

where $\frac{\partial s}{\partial p} > 0$ and $\frac{\partial s}{\partial t} < 0$. We also assume that both the demand and supply functions are continuously differentiable of class C^1.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ by the equation $f(p, t) = d(p) - s(p, t)$ be the excess demand function.

Our set of assumptions is

$$\begin{aligned} &\frac{\partial d}{\partial p} < 0 \\ &\frac{\partial s}{\partial p} > 0 \\ &\frac{\partial s}{\partial t} < 0. \end{aligned} \tag{7.1}$$

Let (p_0, t_0) denote an initial equilibrium, which occurs when demand equals supply and the excess demand is zero, then

$$f(p_0, t_0) = d(p_0) - s(p_0, t_0) = 0.$$

Since we assume $\frac{\partial d}{\partial p} < 0$ and $\frac{\partial s}{\partial p} > 0$, we get

$$\frac{\partial f}{\partial p} = \frac{\partial d}{\partial p} - \frac{\partial s}{\partial p} < 0.$$

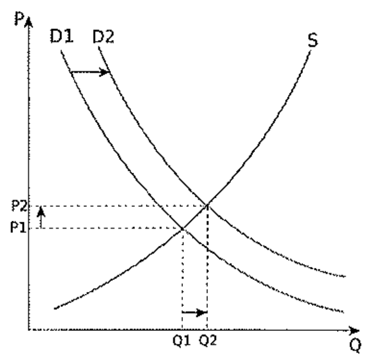
So, by the Implicit Function Theorem, we know that there exists a function g relating the equilibrium market price p to the tax rate t such that $p = g(t)$ in a neighborhood B of t_0. Then, though we cannot derive g explicitly, we can compute its derivative by the formula:

$$\begin{aligned} Dg &= -\left[\frac{\partial f}{\partial p}\right]^{-1} \cdot \frac{\partial f}{\partial t} \\ &= -\left[\frac{\partial d}{\partial p} - \frac{\partial s}{\partial p}\right]^{-1} \left(-\frac{\partial s}{\partial t}\right) \\ &= \frac{\partial s/\partial t}{\partial d/\partial p - \partial s/\partial p}. \end{aligned}$$

Since $\partial s/\partial t < 0, \partial d/\partial p < 0,$ and $\partial s/\partial p > 0$, it follows that $Dg = \frac{\partial p}{\partial t} > 0$, which confirms our economic intuition that an increase in the tax rate would raise the market price.

8. A MARKET MODEL OF TWO SUBSTITUTE GOODS

In any Principle of Economics course, we are taught that in a market of two substitute goods, an increase in price of one good would make the demand for the other good shift outward to the right as shown in the graph below.



Though the graph is good at visualizing the change in equilibrium, we might wonder why and how the demand curve shifts outward when the price of another good increases. Since the graph is in two dimensions, we can only see the price and quantity of one good. Therefore, it is unable to explain the shift in demand given an increase in price of the other good, which would require a third dimension. In this section, we will build a simple model of two substitute goods to explain the demand shift. Moreover, we will examine the effect of relative price ratio on the magnitude of change in equilibrium.

Suppose that in a perfectly competitive car market, we have only two kinds of cars: new cars and used cars. We assume that the demand for used cars is a continuously differentiable function of price of new cars and price of used cars, $Q_{du} = D_u(P_n, P_u)$. It is reasonable to assume that the demand for used cars would increase if the price of used cars decreased, or if the price of new cars increased, i.e. $\frac{\partial D_u}{\partial P_u} < 0, \frac{\partial D_u}{\partial P_n} > 0$.

Moreover, when the price of new cars is at higher level, a fixed amount of an increase in new cars price would make smaller impact on the demand for used cars. For example, a customer would evaluate an increase of \$2,000 for a \$20,000 new car more significant than the same \$2,000 raise for a \$40,000 new car. Therefore, we assume that the effect of change in new car prices on used cars demand is decreasing, i.e. $\frac{\partial^2 D_u}{\partial P_n^2} < 0$.

On the supply side, suppose that the supply for used cars is also a continuously differentiable function of price of new cars and price of used cars, $Q_{su} = S_u(P_n, P_u)$. Since sellers of used cars care about the price of new cars, they would be more willing to sell their cars to upgrade a new one, when the price of new cars decreases. In addition, used car owners would also have more incentive to sell their cars, if the price of used cars increased. Hence, we assume that $\frac{\partial S_u}{\partial P_n} < 0$ and $\frac{\partial S_u}{\partial P_u} > 0$.

Moreover, since used car owners are able to distinguish between relative and absolute change in prices, when the price used cars is at higher level, a fixed amount of an increase in used cars price would make smaller impact on the supply for used cars. Therefore, we assume that $\frac{\partial^2 S_u}{\partial P_u^2} < 0$.

Here is a short summary of assumptions for our model:

$$
\begin{aligned}
\frac{\partial D_u}{\partial P_u}, \frac{\partial S_u}{\partial P_n} &< 0 \\
\frac{\partial D_u}{\partial P_n}, \frac{\partial S_u}{\partial P_u} &> 0 \\
\frac{\partial^2 D_u}{\partial P_n^2} &< 0 \\
\frac{\partial^2 S_u}{\partial P_u^2} &< 0.
\end{aligned}
\tag{8.1}
$$

Considering the model above, we have following theorems

Theorem. *The price of new cars and the price of used cars are positively correlated.*

Proof. Let $f : (\mathbb{R}^+)^2 \to \mathbb{R}^+$ given by $f(P_n, P_u) = D_u(P_n, P_u) - S_u(P_n, P_u)$ be the function of excess demand for used cars. Since functions D_n and S_n are continuously differentiable, so is the function f.

Moreover, since $\frac{\partial D_u}{\partial P_n} > 0, \frac{\partial S_u}{\partial P_n} < 0$, we know $\frac{\partial f}{\partial P_n} = \frac{\partial D_u}{\partial P_n} - \frac{\partial S_u}{\partial P_n} > 0$. Hence, by the Implicit Function Theorem, there exists a function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that $P_n = g(P_u)$ and though we do not explicitly know the function g, we can calculate its derivative by the formula:

$$
\begin{aligned}
\frac{dP_n}{dP_u} = Dg &= -\left[\frac{\partial f}{\partial P_n}\right]^{-1}\left[\frac{\partial f}{\partial P_u}\right] \\
&= -\left[\frac{\partial D_u}{\partial P_n} - \frac{\partial S_u}{\partial P_n}\right]^{-1}\left[\frac{\partial D_u}{\partial P_u} - \frac{\partial S_u}{\partial P_u}\right] \\
&= \frac{\partial S_u/\partial P_u - \partial D_u/\partial P_u}{\partial D_u/\partial P_n - \partial S_u/\partial P_n}.
\end{aligned}
$$

According to our assumption (8.1), $\frac{\partial S_u}{\partial P_u}, \frac{\partial D_u}{\partial P_n} > 0$, $\frac{\partial D_u}{\partial P_u}, \frac{\partial S_u}{\partial P_n} < 0$, it follows that $\frac{dP_n}{dP_u} > 0$. Therefore, our theoretical approach using the Implicit Function Theorem implies that the price of new cars would increase when the price of used car increases, which coincides with the graphical result taught in any Principle of Economics course. □

Theorem. *If the demand and supply functions are linear, the relative price ratio is constant on the equilibrium path.*

Proof. Suppose the demand function for used car is

$$D_u(P_n, P_u) = aP_n - bP_u$$

where a, b are positive. The supply function is

$$S_u(P_n, P_u) = -cP_n + dP_u$$

where c, d are positive.

A market reaches equilibrium when there exists no excess in demand and supply. As a result, to be on the equilibrium path, a set of prices, (P_n, P_u), needs to satisfy the following condition:

$$D_u(P_n, P_u) - S_u(P_n, P_u) = 0,$$

which means

$$aP_n - bP_u + cP_n - dP_u = (a + c)P_n - (b + d)P_n = 0.$$

So, the condition for the market to stay in equilibrium is $\frac{P_n}{P_u} = \frac{b+d}{a+c}$, which is a constant in our simple linear model. Therefore, it does not make sense to estimate the effect of the relative price ratio $\left(\frac{P_n}{P_u}\right)$ on the change in equilibrium. □

Definition. *A function of two independent variables is said to be separable if it can be expressed as a product of two functions, each of them depending on only one variable.*

Example. A function $g : \mathbb{R}^2 \to \mathbb{R}$ given by $g(x, y) = x^3y^7$ is separable, since we can rewrite $g(x, y) = h(x)k(y)$, where $h(x) = x^3$ and $k(y) = y^7$.

Definition. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree k for any $k \in \mathbb{R}$, if*

$$f(tx_1, tx_2, ..., tx_n) = t^k f(x_1, x_2, ..., x_n)$$

for all $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and all $t > 0$.

Example. A function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = xy$ is homogeneous of degree two, since $f(tx, ty) = (tx)(ty) = t^2xy = t^2 f(x, y)$ for all $x, y \in \mathbb{R}$ and all $t > 0$.

Lemma. *A function of one variable is homogeneous of degree t if and only if it can be written in the form $f(x) = ax^t$, where a and t are any real numbers.*

Proof. To prove $f(x) = ax^t$ is homogeneous, let b be any real number. Then, we get

$$\begin{aligned} f(bx) &= a(bx)^t \\ &= b^t(ax^t) \\ &= b^t f(x). \end{aligned}$$

To prove the form $f(x) = ax^t$ is a necessary condition for a function of one variable to be homogeneous, let $f(x)$ be a homogeneous function of degree t, which means for any real number b, $f(bx) = b^t f(x)$, where t is a constant. Consider when $x = 1$, we get $f(b) = b^t f(1) = c \cdot b^t$, where c is a constant. Since b is an arbitrary real number, we conclude that $f(x) = c \cdot x^t$, where c and t are constant. □

Theorem. *If the demand and supply functions are separable and homogeneous, then the relative prices ratio is constant on the equilibrium path if and only if the demand and supply functions have the same degree of homogeneity.*

Proof. Suppose the demand function has the form

$$D_u(P_n, P_u) = h(P_n)k(P_u)$$

And suppose the supply function has the form

$$S_u(P_n, P_u) = p(P_n)q(P_u)$$

Since demand and supply function are always positive, without loss of generality, suppose h, k, p, q are positive functions. Then, by the above lemma, since h, k, p, q are homogenous functions of one variable, we can rewrite our demand and supply functions in a specific form, such as

$$\begin{aligned} D_u(P_n, P_u) &= cP_n^a P_u^b \\ S_u(P_n, P_u) &= dP_n^e P_u^k \end{aligned}$$

where c and d are positive.

Moreover, from the assumption (8.1)

$$\begin{aligned} \frac{\partial D_u}{\partial P_n} &= caP_n^{a-1}P_u^b > 0 \\ \frac{\partial D_u}{\partial P_u} &= cbP_n^a P_u^{b-1} < 0 \\ \text{and} & \\ \frac{\partial S_u}{\partial P_n} &= deP_n^{e-1}P_u^k < 0 \\ \frac{\partial S_u}{\partial P_u} &= dkP_n^e P_u^{k-1} > 0 \end{aligned}$$

we know $a, k > 0$ and $b, e < 0$.

Let $\alpha(P_n, P_u) = \dfrac{P_n}{P_u}$ be the function of the relative price ratio. If $\alpha(P_n, P_u)$ is a constant, then substitute $P_n = \alpha P_u$ in our market equilibrium condition, $D_u = S_u$, we get

$$\begin{aligned} c(\alpha P_u)^a P_u^b &= d(\alpha P_u)^e P_u^k \\ c\alpha^a P_u^{a+b} &= d\alpha^e P_u^{e+k} \\ \alpha^{a-e} &= \frac{d}{c} P_u^{(e+k)-(a+b)} \end{aligned}$$

Since α is a constant, it follows that $(e + k) - (a + b) = 0$, or equivalently, $e + k = a + b$. Hence, the demand and supply functions have the same degree of homogeneity.

Conversely, suppose the demand and supply functions have the same degree of homogeneity, which means $e + k = a + b$.

Then, our market equilibrium condition $D_u = S_u$ implies

$$\begin{aligned} cP_n^a P_u^b &= dP_n^e P_u^k \\ P_n^{a-e} &= \frac{d}{c} P_u^{k-b} \\ \frac{P_n^{a-e}}{P_u^{a-e}} &= \frac{d}{c} P_u^{(k-b)-(a-e)} \\ \alpha^{a-e} &= \frac{d}{c} P_u^{(k+e)-(a+b)} \end{aligned}$$

Since $(k + e) - (a + b) = 0$, it follows that $\frac{d}{c} P_u^{(k+e)-(a+b)}$ is constant, and so is α. □

Theorem. *If the demand and supply functions are separable and homogeneous of opposite degrees, then the relative prices ratio has an impact on the effect of changes in prices on quantity equilibrium.*

Proof. Recall our model

$$\begin{aligned} D_u(P_n, P_u) &= cP_n^a P_u^b \\ S_u(P_n, P_u) &= dP_n^e P_u^k \end{aligned}$$

where $a, k, c, d > 0$, and $b, e < 0$.

Then, the condition for the market equilibrium $D_u = S_u$ means $cP_n^a P_u^b = dP_n^e P_u^k$, or equivalently,

$$P_n^{a-e} = \frac{d}{c} P_u^{k-b}.$$

Our goal is to estimate $\frac{d(\partial D_u/\partial P_u)}{d(P_n/P_u)}$ and $\frac{d(\partial D_u/\partial P_n)}{d(P_n/P_u)}$. For both derivatives, we will try to use the chain rule, i.e

$$\frac{d(\partial D_u/\partial P_u)}{d(P_n/P_u)} = \frac{d(\partial D_u/\partial P_u)}{dP_u} \cdot \frac{dP_u}{d(P_n/P_u)} \tag{8.2}$$

$$\frac{d(\partial D_u/\partial P_n)}{d(P_n/P_u)} = \frac{d(\partial D_u/\partial P_n)}{dP_u} \cdot \frac{dP_u}{d(P_n/P_u)} \tag{8.3}$$

To solve (8.2), we will try to derive $\partial D_u/\partial P_u$ explicitly as a function of P_u, and P_u as a function of $\alpha = \frac{P_n}{P_u}$ using the market equilibrium condition

$$P_n = (\frac{d}{c} P_u^{k-b})^{\frac{1}{a-e}} \tag{8.4}$$

1. Substitute P_n from the equation 8.4 into $\frac{\partial D_u}{\partial P_u}$, we get

$$\begin{aligned} \frac{\partial D_u}{\partial P_u} &= cbP_n^a P_u^{b-1} \\ &= cb(\frac{d}{c} P_u^{k-b})^{\frac{a}{a-e}} P_u^{b-1} \\ &= cb(\frac{d}{c})^{\frac{a}{a-e}} P_u^{\frac{(k-b)\cdot a + (b-1)(a-e)}{a-e}} \\ &= cb(\frac{d}{c})^{\frac{a}{a-e}} P_u^{\frac{ka-ba+ba-a-be+e}{a-e}} \\ &= cb(\frac{d}{c})^{\frac{a}{a-e}} P_u^{\frac{ka-a-be+e}{a-e}} \end{aligned}$$

Then,

$$\frac{d(\partial D_u/\partial P_u)}{dP_u} = cb(\frac{d}{c})^{\frac{a}{a-e}} \frac{ka-a-be+e}{a-e} P_u^{\frac{ka-a-be+e}{a-e}-1}$$

Since we assume that the effect of change in used car prices on supply is decreasing, i.e. $\frac{\partial^2 S_u}{\partial P_u^2} = dk(k-1)P_n^e P_u^{k-2} < 0$, it follows that $0 < k < 1$ since d, k, P_n, P_u are positive. Thus, $ka - a = a(k-1) < 0$.

Moreover, since $b, e < 0$, and $a, k, c, d > 0$, we know $-be < 0, e < 0$, and $a - e > 0$, so that $\frac{ka-a-be+e}{a-e} < 0$. Hence, since $cb(\frac{d}{c})^{\frac{a}{a-e}} < 0$ and $P_u > 0$, it follows that

$$\frac{d(\partial D_u/\partial P_u)}{dP_u} > 0. \tag{8.5}$$

At this point, we have solved the first part, $\dfrac{d(\partial D_u/\partial P_u)}{dP_u}$, in our chain rule formula; the second part, $\dfrac{dP_u}{d(P_n/P_u)}$, will be examined below.

2. From the market equilibrium condition 8.4, $P_n^{a-e} = \dfrac{d}{c}P_u^{k-b}$, we can derive P_u as a function of $\dfrac{P_n}{P_u} = \alpha$ as below

$$\begin{aligned}
\frac{c}{d} \cdot \frac{P_n^{a-e}}{P_u^{a-e}} &= P_u^{k-b}.P_u^{-(a-e)} \\
\Rightarrow \frac{c}{d}(\alpha)^{a-e} &= P_u^{k-b-a+e} \\
\Rightarrow \frac{c}{d}(\alpha)^{\frac{a-e}{k-b-a+e}} &= P_u.
\end{aligned}$$

Thus, the derivative of P_u with respect to α

$$\begin{aligned}
\frac{dP_u}{d\alpha} &= \frac{c}{d} \cdot \frac{a-e}{k-b-a+e} \cdot \alpha^{\frac{a-e}{k-b-a+e}-1} \\
&= \frac{c}{d} \cdot \frac{a-e}{(k+e)-(a+b)} \cdot \alpha^{\frac{a-e}{k-b-a+e}-1}.
\end{aligned}$$

Since the demand of supply functions are homogeneous of opposite degrees, there exists two cases:

1. If the supply function is homogenous of positive degree, $(k+e) > 0$, then $(a+b) < 0$, and so $(k+e) - (a+b) > 0$.

As a result, since $c, d, \alpha > 0$, $\dfrac{a-e}{(k+e)-(a+b)} > 0$ implies

$$\frac{dP_u}{d\alpha} > 0. \tag{8.6}$$

2. If the supply function is homogenous of negative degree, $(k+e) < 0$, then $(a+b) > 0$, and so $(k+e) - (a+b) < 0$.

Thus,

$$\frac{dP_u}{d\alpha} < 0. \tag{8.7}$$

From 8.2, 8.5, 8.6, and 8.7, it follows that $\dfrac{d(\partial D_u/\partial P_u)}{d\alpha} = \dfrac{d(\partial D_u/\partial P_u)}{dP_u} \cdot \dfrac{dP_u}{d\alpha}$ is positive if the supply function is homogenous of positive degree; and vice-versa, $\dfrac{d(\partial D_u/\partial P_u)}{d\alpha}$ is negative if the supply function is homogenous of negative degree.

Similarly, we want to estimate the equation 8.3

$$\frac{d(\partial D_u/\partial P_n)}{d(P_n/P_u)} = \frac{d(\partial D_u/\partial P_n)}{dP_u} \cdot \frac{dP_u}{d(P_n/P_u)}$$

Since we have already examined $\dfrac{dP_u}{d\alpha}$ in equations 8.6 and 8.7, all we need to do is solve for $\dfrac{d(\partial D_u/\partial P_n)}{dP_u}$.

Recall that $\frac{\partial D_u}{\partial P_n} = caP_n^{a-1}P_u^b$, then we substitute $P_n = (\frac{d}{c}P_u^{k-b})^{\frac{1}{a-e}}$ in equation 8.4 into the function and get

$$\begin{aligned}\frac{\partial D_u}{\partial P_n} &= caP_n^{a-1}P_u^b \\ &= ca(\frac{d}{c}P_u^{k-b})^{\frac{a-1}{a-e}}P_u^b \\ &= ca(\frac{d}{c})^{\frac{a-1}{a-e}}P_u^{\frac{(k-b)(a-1)+b(a-e)}{a-e}} \\ &= ca(\frac{d}{c})^{\frac{a-1}{a-e}}P_u^{\frac{ka-ba-k+b+ba-be}{a-e}} \\ &= ca(\frac{d}{c})^{\frac{a-1}{a-e}}P_u^{\frac{ka-k+b-be}{a-e}}.\end{aligned}$$

The derivative of $\frac{\partial D_u}{\partial P_n}$ with respect to P_u

$$\frac{d(\partial D_u/\partial P_n)}{dP_u} = ca(\frac{d}{c})^{\frac{a-1}{a-e}}\frac{ka-k+b-be}{a-e}P_u^{\frac{ka-k+b-be}{a-e}-1}.$$

Since we assume the effect of change in new car prices on demand is decreasing, we know $\frac{\partial^2 D_u}{\partial P_n} = ca(a-1)P_n^{a-1}P_u^b < 0$.

Hence, since $c, a, P_n, P_u > 0$, it follows that $a < 1$. Moreover, since $b.e < 0$, $b < 0$, $-be < 0$ and $a-e > 0$, we get $\frac{ka-k+b-be}{a-e} < 0$ and $\frac{d(\partial D_u/\partial P_n)}{dP_u} < 0$ (since $ca(\frac{d}{c})$ and P_u are positive).

As a result, $\frac{d(\partial D_u/\partial P_n)}{d(\alpha)}$ is positive if the supply function is homogenous of negative degree, and is negative if the supply function is homogenous of positive degree. □

9. Discussion

The Implicit Function Theorem is extremely helpful and widely used in economics literature since when setting up mathematical models in economics, we are often faced equations which cannot be solved to get endogenous variables as explicit functions of exogenous variables and parameters. However, it also has some drawbacks. First, our economic functions must be continuously differentiable to apply the Implicit Function Theorem, which is a quite strong assumption. Second, by using the Implicit Function Theorem, our results are only guaranteed for a small local neighborhood around the equilibrium, so that it is not adequate to predict effects of a huge economic shock. Third and most importantly, the Implicit Function Theorem and comparative statics analysis ignore the dynamic nature of economics, so that beside the estimated effect of a change in exogenous variables, we barely know the mechanism of that change, which is the focus of economic theory.

In section 8, we attempt to go beyond the application of the Implicit Function Theorem to find a factor that can influence the magnitude of the change in equilibrium. Although the relative price ratio is a promising candidate, the result in section 8 is limited due to the assumptions of our model (homogeneity, separability, and concavity). We acknowledge that a more general result would require a stronger economic foundation and a more complexed approach (e.g. dynamic general equilibrium model).

10. Conclusion

Section 1 introduces a simple market model of one good that we can solve explicitly. However, because we want to analyze the change of equilibrium in more complicated economic models, we need a stronger mathematical tools, such as the Implicit Function Theorem. A full proof of the Implicit Function Theorem is presented in section 4. Basically, the Theorem consists of two main parts: conditions (continuity and non-singularity) for the existence and uniqueness of the implicit function, and the formula for its derivative. The proof technique requires a broad background in mathematics, as we need to use multivariable calculus, linear algebra, and basic point-set topology. The Implicit Function Theorem is widely used in economics because it demands only two reasonable conditions but produces a powerful results.

Section 5 and 7 exhibit two examples to show how we use the Implicit Function Theorem in analyzing economic models. The procedures of our analysis are quite similar for both IS - LM model and the market model with tax parameter. First, we set up a simple function that represents our model. Then, based on real-life facts, we create a set of restrictions and assumptions for our function. After verifying that the function satisfies all required conditions, we apply the formula from the Implicit Function Theorem to draw conclusions.

Section 8 analyzes the impact of the relative price ratio on the magnitude of change in equilibrium in a market model of two substitute goods. Our analysis concludes that the relative price ratio has a positive impact on the effect of changes in used car price on the quantity demanded for used cars if the supply function has a positive and the demand function has a negative degree of homogeneity. Vice-versa, the relative price ratio has a negative impact if the supply function has a negative and the demand function has a positive degree of homogeneity. With a similar approach, our model also proves that the relative price ratio has a positive impact on the effect of changes in new car price on the quantity demanded for used cars if the supply has a negative and the demand has a positive degree of homogeneity, and vice-versa.

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