A GEOMETRIC ANALYSIS OF CHORD PROGRESSIONS

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ABSTRACT. Dmitri Tymoczko and others have applied mathematical techniques to music theoretical analysis. We followed Tymoczko's set notation in our analysis. Our approach seeks to define a metric for finding minimal voice leadings between two given chords. We use our metric to examine all possible leadings and visualize the results with graphical representations. We use two-note chords rendered as points in a Mobius strip to examine their voice leading relationships. We then extend this analysis to three-note chords in a torus.

1. INTRODUCTION

The tonnetz is a classical geometric representation of chords. Introduced by Euclid, it is a lattice structure that plots notes to show perfect fifths, minor thirds, and major thirds near each other, and makes it possible to see major and minor triads.

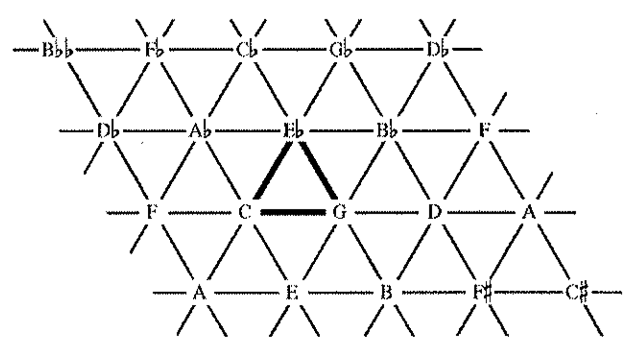


FIGURE 1. The tonnetz has perfect fifths arranged along the horizontal lines, C to G for instance, minor thirds along the diagonals rising from left to right, C to E^b, and major thirds along the diagonals descending from left to right, E^b to G. This arrangement groups major and minor triads in triangles; all of the triangles with a vertex pointing downward are major triads, such as CEG, and all of the triangles with upward facing vertices are minor triads, such as the highlighted CE^bG [5].

Other methods of determining the closeness of chords consider aspects of acoustic consonance besides the three relationships considered in the tonnetz. These methods are based on psychological studies that have determined which intervals sound most consonant, or most harmonically pleasing. Geometric structures, such as the tonnetz, are "calibrated" to position perfect fifths, the most consonant sounding intervals, nearest each other, and a diminished fifth, the least consonant, furthest[3, p. 2]. Other geometric spaces have also been developed to describe closeness. For instance, Fred Lerdahl defines tonal pitch space, which incorporates the chord and the key it is in to determine the amounts of tension present in the chord[3, p. 2].

Tymoczko takes an approach based on logarithmic pitch values to determine the closeness of chords. He claims that there exist five qualities that are characteristic of tonality throughout all cultures in the development of music: conjunct melodic motion, acoustic consonance, harmonic consistency, limited macroharmony, and centricity [4, p. 4]. A conjunct melodic motion corresponds to small voice leading distances. Acoustic consonance refers to the preference of consonant harmonies over disonant ones. Harmonic consistency means that the harmonic structures tend to be similar throughout a piece (example: use of major chords, minor chords, etc.) Macroharmony involves the number of pitches typically used in a piece, which relates directly to the key in which the piece is written and is typically between five and eight notes. Finally, centricity means that one pitch is heard more prominently than others and typically appears more frequently, acting as a "destination" pitch [4, p. 4]. Our work is based on the system Tymoczko uses to analyze these features. We focus on the use of conjunct melodic motion by minimizing voice leading sizes.

2. Background

There are three basic musical objects we are dealing with and defining in this section. The first of which are pitch sets. Pitches are represented by numbers based on their logarithmic frequency. Allowing middle C (or C4, the C occurring in the 4th octave) to equal 0, we then number each equal tempered pitch with integers based on its relationship to middle C. This makes D4 = 2 because it is two semitones above C4, and C3 = -12 because it is 12 semitones below C4. We then create sets to represent multiple pitches performed simultaneously.

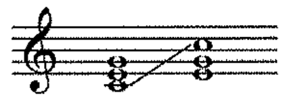
Definition. A pitch multiset is an ordered n-tuple $(v_1, v_2, ..., v_n)$ that represents n pitches sounding simultaneously, where v_n is the note performed by the nth voice.

Example. Thus, if a saxophone plays D4, a trumpet plays F4, and a flute plays A4, the resulting pitch set is $(2, 6, 9)$. Since order matters, when the notes changes, ex. $(2,6,9) \to (7,11,14)$, each instrument plays the corresponding note; in this case each player is five semitones higher than their previous note.

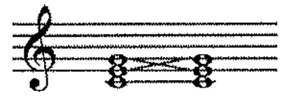
We consider five operations that are performed on a set of pitches to change it. They are Octave shifts, Permutations, Transpostitions, Inversions, and Cardinality changes - OPTIC [1]. Octave shifts move one note of the set up or down by 12 semitones. Permutations rearrange the order of the set, changing the note performed by each voice. Transpositions move all notes up by the same amount. Inversions flip the set around a pivot pitch (different from the musical term inversion, which refers to the different orderings of notes in a chord). Cardinality changes add or remove doubled pitches. These manipulations will be used to abstract to different spaces.

Example. The following are examples of OPTIC transformations [2].

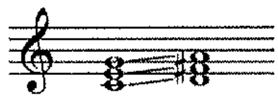
Octave Shift



Permutation

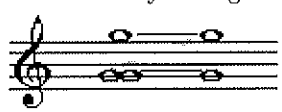


Transposition



Inversion

Cardinality Change



The first level of abstraction from pitch space is pitch class space, in which arithmetic is modulo 12, creating octave equivalence. This corresponds to modding out by octave shifts. This lets all Cs be equal to 0 and all Ds be equal to 2 and so on.

Definition. A pitch class multiset is an ordered n-tuple $(v_1, v_2, ..., v_n)$ that represents n pitches sounding simultaneously, where v_n is the note performed by the nth voice, mod12.

Example. When the pitch sets from the previous example are examined under pitch class notation, $(2,6,9) \to (7,11,14)$ becomes $(2,6,9) \to (7,11,2)$. The order still matters, so $2 \to 7$, $6 \to 11$, and $9 \to 2$(since $14 \bmod 12 \equiv 2$).

We next mod out by permutations, resulting in chord set notation, our next level of abstraction. Here we distinguish between the musical definition of a chord and our mathematical definition. We care only that the chord is a set of simultaneous pitch classes and remove any other restrictions on the defintion.

Definition. A chord multiset is an unordered set $\{v_1, v_2, ..., v_n\}$ that represents n pitches sounding simultaneously, mod12.

At this level of abstraction, order no longer matters and we have chord equivalence. When considering chords, the pitch sets $(2,6,9) = (2,9,6) = (6,2,9) = (6,9,2) = (9,2,6) = (9,6,2)$. All of these pitch sets are represented by an unordered chord set such as $\{2,6,9\}$. Also, contrary to some musical analysis, we are not able to distinguish between chord inversions. So, for us a C major chord in root position, (C,E,G), is the same as a C major chord in first inversion, (E,G,C).

We will examine the voice leading relationships among the musical objects (pitch sets, pitch class sets, chord sets) defined above. Voice leadings are the transitions between individual notes from one chord to the next. We want to minimize the combined total movement of notes when chords transition. The transitioning of chords is referred to as chord progressions. The idea of "conjunct melodic motion" expresses the preference for smaller voice leading distances. Another aspect of chord progression involved in Western music is the avoidance of voice crossings. A voice crossing occurs when the voice performing a pitch lower than another voice moves to a pitch higher than the other voice.

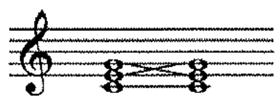


FIGURE 2. While the pitches stay the same, the voices performing them change positions creating a voice crossing.

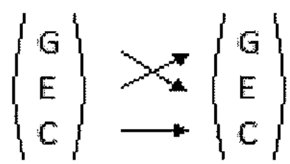


FIGURE 3. We can also express the previous example in the form of pitch sets. The arrows represent voice leadings, so the middle voice on E moves up to G, and the high voice on G moves down to E, creating a crossing.

With these two "rules" in mind, our goal is to evaluate the relationships between any two chord sets to find the minimal voice leading distance with no voice crossings.

3. Two-Note Chords

We begin by examining two-note sets. At each level of abstraction, the space of the musical objects is altered. We illustrate these spaces and representations of our musical objects. We treat the sets as Cartesian coordinates, resulting in ordered pairs of pitches, which are then plotted.

Voice leadings are visible in these spaces. The voice leading between any two musical objects is represented by a line drawn between them. While all of the sets we have contain integer values, the following analysis can be thought of in terms of real numbers. Thus, voice leadings can be conceptualized as the sliding of each pitch to the next along this line. Since we hope to minimize the lengths of voice leadings, we will minimize the lengths of the lines representing them.

Example. In the voice leading from $(1,2) \to (3,4)$, 1 slides to 3, passing over all real values between 1 and 3, and 2 slides to 4, passing over all real values between 2 and 4.

The distance between two pitch sets is easily defined using the Euclidean, or standard, metric, $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, since the points are fixed. Pitch class sets and chord sets correspond to an infinite number of points in pitch space so distance is less easliy defined. We define a distance in these spaces by fixing the starting chord and finding the closest representative of the ending chord.

3.1. **Geometric Spaces.** We begin with pitch sets, which exist in pitch space.

Theorem 3.1. *Pitch space is equivalent to the infinite plane* $\mathbb{Z} \times \mathbb{Z}$.

Proof. To show equivalence, we show that there exists a one-to-one correspondence between the two spaces.

Let (p_1, p_2) be a pair of equal tempered pitches in pitch space and $(a, b) \in \mathbb{Z} \times \mathbb{Z}$. Let C4=0. Then (p_1, p_2) maps to (a, b) if

C4+a semitones= $a = p_1$

and

C4+b semitones= $b = p_2$.

This is one-to-one because every (p_1, p_2) maps to a unique (a, b). □

In pitch space, we observe voice leadings and voice crossings as they would appear in performance. Voice crossings occur along the line $y = x$. This line represents both voices sounding unison pitches. A voice leading that passes over this line contains a voice crossing.

Example. We can graph voice leadings in $\mathbb{Z} \times \mathbb{Z}$ to visualize them.

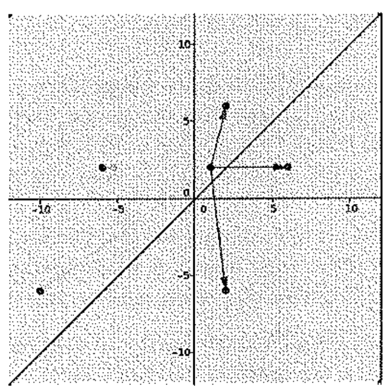


FIGURE 4. Here are five leadings from $(1, 2) \to \{2, 6\}$. We represent $\{2, 6\}$ by the ordered pairs $(2, 6), (6, 2), (-6, 2), (2, -6)$ and $(-10, -6)$. We see the black and dark blue leadings contain voice crossings because they intersect $y = x$.

Transitioning to pitch class space from pitch space is simply removing all copies of notes that are equivalent by the octave shift. This reduces the infinite plane to the 12 by 12 tile.

Theorem 3.2. *Pitch class space is equivalent to* $\mathbb{Z}_{12} \times \mathbb{Z}_{12}$.

Proof. We can write a one-to-one correspondence between the spaces. There are twelve pitch classes, {C, C#, D, ... , B}, which we match up with the integers of $\mathbb{Z}_{12}$ by letting C=0,C#=1, ... , B=11. Thus, we can take (p_1, p_2) from pitch class space and map it to a unique $(a, b) \in \mathbb{Z}_{12} \times \mathbb{Z}_{12}$. ☐

Example. This example shows the same five leadings from $(1,2) \rightarrow \{2,6\}$ as the previous example.

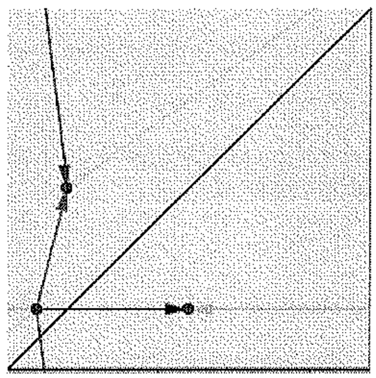


FIGURE 5. The black and dark blue arrows still contain crossings. The dark blue, light blue, and green arrows show behavior in pitch class space when a voice leading passes into another octave. Also note that all five representations of {2,6} are described by the two points (2,6) and (6,2).

There are infinite paths between each pair of sets since each pair represents an infinite number of points in pitch space. This is because each pitch class set, (x, y), corresponds to all of the pitch sets $(x + 12m, y + 12n)$ for $m, n \in \mathbb{Z}$. There are an infinite amount of these pitch set representations, and the voice leadings differ for each.

This depiction reveals that voice crossings are less readily observed in pitch class space. After abstraction to pitch class space, a voice leading that passes over the line $y = x$ does not necessarily mean the voices had to cross in pitch space.

Example. For instance, if a piccolo drops down two octaves from G6 to G4 and a tuba moves up from D3 to D4, the pitch sets do not have a crossing, but the pitch class sets do.

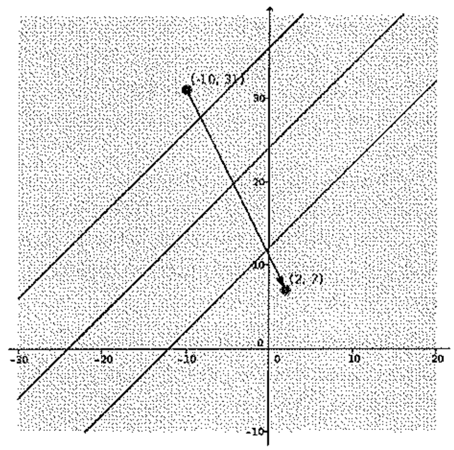


FIGURE 6. We see the pitch sets as $(-10, 31) \rightarrow (2, 7)$. In this case, the voice leading passes over the lines $y = x + 12$, $y = x + 24$, and $y = x + 36$, all of which become $y = x$ when we are mod12.

However, this line, $y = x$, still holds significance, particularly when moving between chord space and pitch class space.

Example. Each (non unision) chord set has two representations in pitch class space. The line $y = x$ separates the two representations of each chord symmetrically.

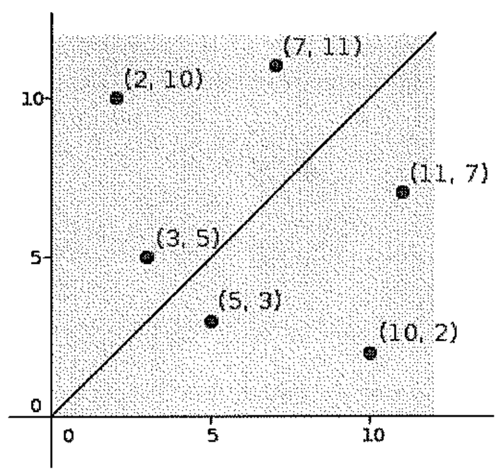


FIGURE 7. We see three examples of pitch sets reflected across the line $y = x$. The coordinates of these pitch sets are congruent to their reflections in chord space.

Theorem 3.3. *Chord space is equivalent to the Mobius strip* $\mathbb{T}^2/S_2$.

Proof. We can create a one-to-one correspondence between the two spaces.

We write a correspondence, $\{p_1, p_2\} \to \mathbb{Z}_{12} \times \mathbb{Z}_{12}$. We see $\mathbb{Z}_{12} \times \mathbb{Z}_{12}$ is topologically equivalent to the torus, $\mathbb{T}^2$, by identifying opposite sides.

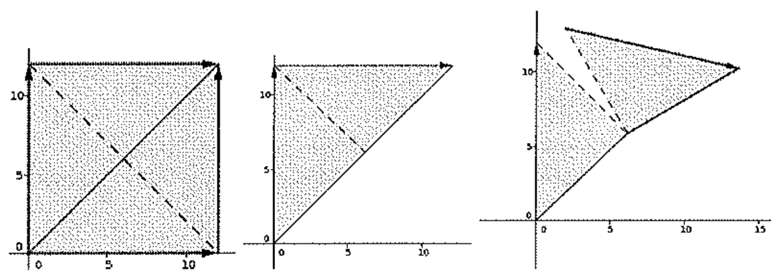
But, this correspondence is not one-to-one; it is one-to-two because each chord is mapped to two ordered pairs.

To make it one-to-one we mod out by permutations on a set of two, or S_2. The simplest way to do this is to take half of the torus. We can then rearrange it into a Mobius strip (see figure 8). The Mobius strip puts chords close to each other that we would expect to be close by wrapping around the edges.

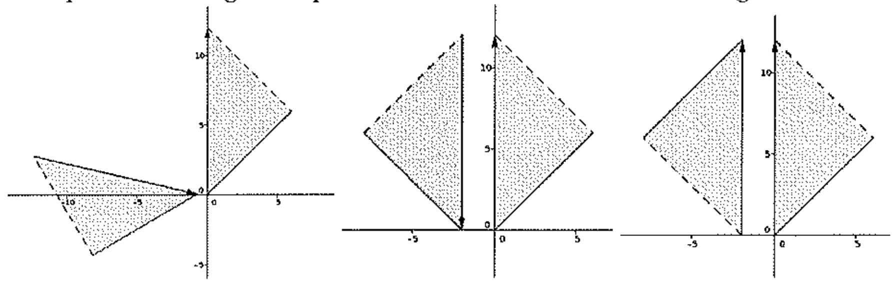
□

We can visualize the transformation from pitch class space to chord space. Since pitch class space contains two copies of each chord, to transform it into chord space, we must remove half of the values. We consider the top half of the pitch class tile, divided along the unison line. We rearrange this half into the Mobius strip for chord space.

Example. I have marked the sides of the tile with arrows showing which way the values are increasing from $0 \to 12$.



First, make an arbitrary cut along $y = -x$ (the dashed line). Take half of the space containing one representative of each chord. Break along the cut.



Then, shift the top piece over to the bottom left. Rotate it so the directed sides are parallel. Flip it so the directed sides run in the same direction.
The final step is to glue the pieces together along the directed sides. One comment to avoid confusion: the solid edges of the final Mobius strip here represent the unisons, which will be represented by a dashed line in the following sections (see figure 9).

FIGURE 8. We transform pitch class space into chord space by modding out permutations and arranging it into a Mobius strip.

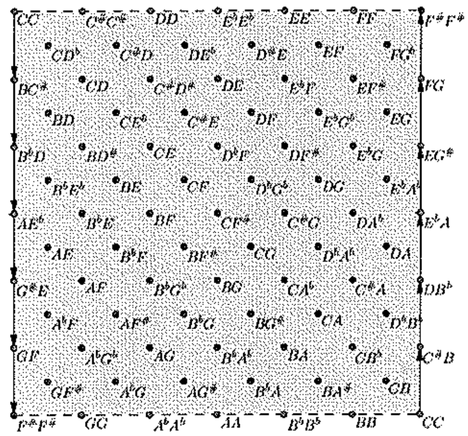


FIGURE 9. The Mobius strip contains one representative of each chord. The left and right sides in this figure are glued together with a 180 degree twist.

3.2. **Behavior in Chord Space.** Tymoczko uses a Mobius strip as the unordered chord space in two dimensions [4, p. 70]. The chords within the Mobius strip are arranged so that unisons, where both pitches are the same, are along the edge. Moving away from the edge moves the pitches further apart and the chords with pitches furthest from each other are along the center. This arrangement places nearby chords close together visually (see figure 9.)

In the representation of the Mobius strip that we use, there appear to be two boudaries. The solid edges are an arbitrary "cut" that are connected to each other with a 180 degree twist. Therefore, a voice leading that passes off a solid edge reenters from the other edge and continues as before. The dashed edges act as "bounce" edges for voice leadings. Since they form the boundary of the Mobius strip and contain all the unison pitches, approaching a dashed edge moves the pitches closer together until they are identical, and then bouncing off moves them apart after they cross.

Example. We see similar examples plotted in pitch class space and chord space.

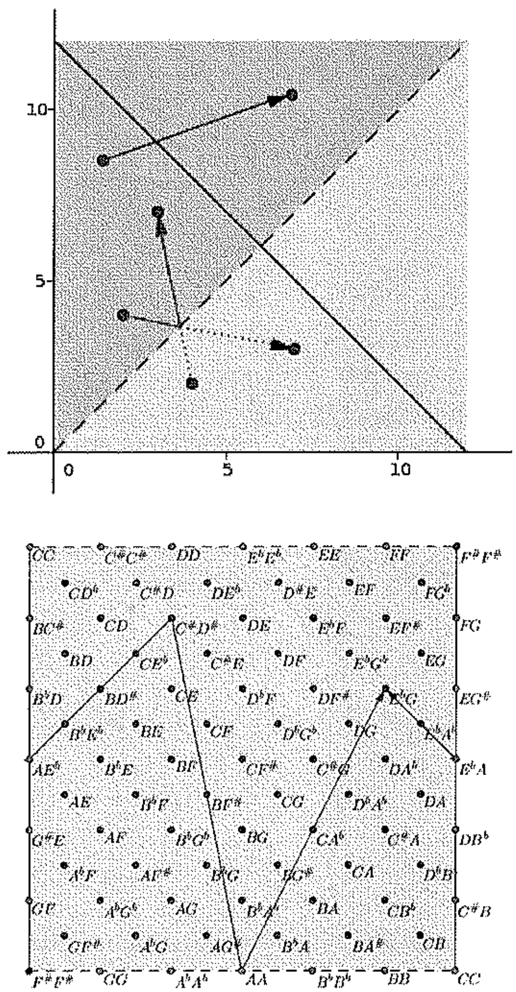


Figure 10. The blue lines show a "bounce" edge, the black lines show a "wrap" edge. In pitch class space, the blue line passes over the voice crossing line $y = x$, and the "bounce" is a result of observing two different voice leadings only while they exist in one half of the tile. The black line remains in the half-tile, so the wrapping effect is a result of cutting along $y = -x$ and gluing it to the other side with a flip.

There exist an infinite number of voice leading paths from one chord to another. Since the "bounce" edges result in voice crossings, we would like to avoid paths that hit them. We would also like to minimize the length of the path we pick between them.

Definition. We define the distance between two two-note chords to be the minimum of all possible paths between them.

Example. Four paths in chord space show different voice leadings from $C^\# D^\#$ to $E^b G$.

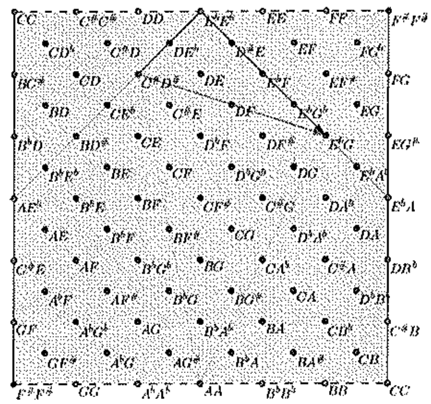


FIGURE 11. The distance from $C^\# D^\#$ to $E^b G$ is the minimum of all paths between the two chords (including those not shown).

Taking a different path in the Mobius strip corresponds to choosing a different representative in pitch space. In order to find the distance in chord space, we pull the pitches back into pitch space and are able to identify a shortest path. It is necessary to use pitch space because the information lost when modding out by the octave eliminates several representations of the chord.

Example. The same paths in the previous example map to different representatives in pitch space.

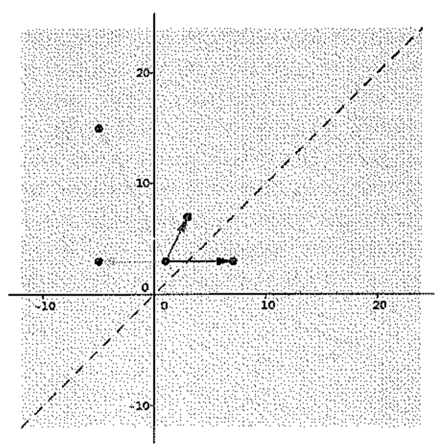


FIGURE 12. We see that each path in pitch class space corresponds to a different representation in pitch space.

3.3. **Measuring Chord Distances.** We use Euclidean distance to determine the length of the voice leadings $(a,b) \to (x,y)$ in pitch space. Since pitch space has translational symmetry, we observe that all of the minimal voice leading paths in chord space occur infinitely many times in pitch space.

Example. We see five identical voice leading paths occuring in two octaves. These same paths occur in every octave so one octave is sufficient to give us all of possible voice leading lengths.

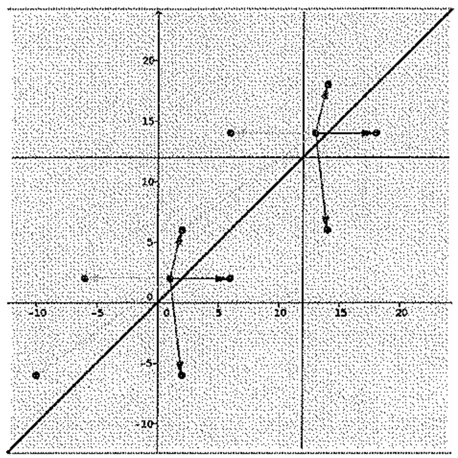


FIGURE 13. We have equivalent voice leading distances from any representation of the starting chord.

To remove the repetitions of congruent paths, we restrict the starting chord $\{a,b\}$ to exist in one copy of chord space, so let $a \le b$ for $(a,b) \in \mathbb{Z}_{12} \times \mathbb{Z}_{12}$.

Since there exist an infinite number of congruent pitch sets for each ending chord, we will categorically eliminate certain voice leading pathways that will never produce a minimum voice leading. To do this, we characterize the ending chord sets $\{x,y\}$ by applying the OPTIC operations. First, examine the permutations, $(a,b) \to (x,y)$ for $x \le y$ and $(x,y) \in \mathbb{Z}_{12} \times \mathbb{Z}_{12}$ and $(a,b) \to (y,x)$ for $x \le y$ and $(y,x) \in \mathbb{Z}_{12} \times \mathbb{Z}_{12}$, which gives us the two pitch class representations of the chord. Then, shift the octaves of the pitch class representations by adding multiples of twelve to either coordinate, resulting in every congruent pitch set.

We will examine each pitch class representation individually; first let $x \le y$ for $(x,y) \in \mathbb{Z}_{12} \times \mathbb{Z}_{12}$. Then there are 9 ways to move the pitches while maintaining the same chord:

(1) $(a,b) \to (x,y)$
(2) $(a,b) \to (x-12m,y)$
(3) $(a,b) \to (x+12m,y)$
(4) $(a,b) \to (x,y-12m)$
(5) $(a,b) \to (x,y+12m)$
(6) $(a,b) \to (x-12m,y+12n)$
(7) $(a,b) \to (x+12m,y-12n)$
(8) $(a,b) \to (x+12m,y+12n)$

(9) $(a,b) \to (x-12m, y-12n)$
where $m,n \in \mathbb{N}$.

Similarly, in the case where we let $x \le y$ for $(y,x) \in \mathbb{Z}_{12} \times \mathbb{Z}_{12}$, we move the pitch sets in the same manner to attain an additional 9 paths:

(10) $(a,b) \to (y,x)$
(11) $(a,b) \to (y-12m, x)$
(12) $(a,b) \to (y+12m, x)$
(13) $(a,b) \to (y, x-12m)$
(14) $(a,b) \to (y, x+12m)$
(15) $(a,b) \to (y-12m, x+12n)$
(16) $(a,b) \to (y+12m, x-12n)$
(17) $(a,b) \to (y+12m, x+12n)$
(18) $(a,b) \to (y-12m, x-12n)$

where $m,n \in \mathbb{N}$.

We next use these path structures to eliminate the voice leadings that are never minimal.

3.4. **Minimal Voice Leadings.** Each of the 18 path structures brings the ending chord to a different half-tile in pitch space. The paths still produce an infinite number of ending pitch sets since $m,n \in \mathbb{N}$. These paths are narrowed further to a finite set of paths when seeking minimal voice leading distances:

Theorem 3.4. *There exist only five paths in pitch space that can result in a minimal distance between two-note chords:*
For $x \le y$:

$$(a,b) \to (x,y)$$
$$(a,b) \to (x-12, y-12)$$
$$(a,b) \to (x+12, y+12)$$
$$(a,b) \to (y-12, x)$$
$$(a,b) \to (y, x+12)$$

These paths correspond to the structures (1),(8),(9),(11), and (14) above, with $m = n = 1$. It is intuitive that moving the ending pitches more than one octave would create a greater distance than moving the ending pitch one or zero octaves.

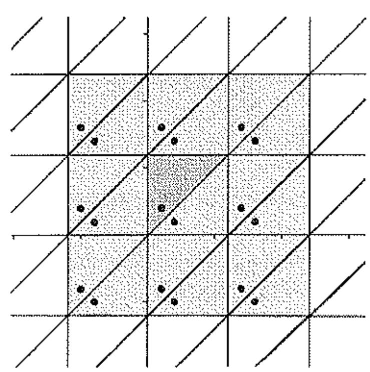


FIGURE 14. The starting chords are fixed in the blue tile. The other colored tiles show a radius of one octave.

Lemma 3.1. *Shifting either pitch two or more octaves creates a larger voice leading distance than at least one of the above five paths.*

Proof. Examine the distances for $(a, b) \to (x, y)$ and $(a, b) \to (x + 12m, y + 12n)$ for $m, n \in \mathbb{Z}$, which concisely represents all equivalent pitch sets. Assume $d((a, b), (x, y)) > d((a, b), (x + 12m, y + 12n))$. Then we have

$$\sqrt{(a-x)^2 + (b-y)^2} > \sqrt{(a - x - 12m)^2 + (b - y - 12n)^2}$$

When we drop the square roots, we retain the inequality because the numbers under them are always positive.

$$(a-x)^2 + (b-y)^2 > (a - x - 12m)^2 + (b - y - 12n)^2$$

Next, we expand each component.

$$a^2 - 2ax + x^2 + b^2 - 2by + y^2 >$$
$$a^2+x^2+(12m)^2-2ax-2(12m)a+2(12m)x+b^2+y^2+(12n)^2-2by-2(12n)b+2(12n)y$$

We move all terms to the right side of the inequality,

$$0 > (12m)^2 - 2(12m)a + 2(12m)x + (12n)^2 - 2(12n)b + 2(12n)y$$

then group the terms

$$0 > 12m^2 - 2ma + 2mx + 12n^2 - 2nb + 2ny$$

and factor.

$$0 > 12m^2 + 2m(x-a) + 12n^2 + 2n(y-b)$$

We now examine several cases. In each case, we will show by contradiction that one of the paths given in Theorem 3.4 is always better than one of the paths that has a two octave shift.

We know $-12 < (x-a) < 12$ and $-12 < (y-b) < 12$. Consider the inequality

$$0 > 12m^2 + 2m(x-a) + 12n^2 + 2n(y-b)$$

When we plug in values for m, only $12m^2 + 2m(x-a)$ is affected, and when we plug in values for n, only $12n^2 + 2n(y-b)$ is affected.

- If $m \geq 2$ then $12m^2 + 2m(x-a) \geq 48 + 4(x-a) \geq 0$
- If $n \geq 2$ then $12n^2 + 2n(y-b) \geq 48 + 4(y-b) \geq 0$
- If $m \leq -2$ then $12m^2 + 2m(x-a) \geq 48 - 4(x-a) \geq 0$
- If $n \leq -2$ then $12n^2 + 2n(y-b) \geq 48 - 4(y-b) \geq 0$

Since each of these is non-negative, the sum of two is non-negative. So we have a contradiction. Therefore the path from $(a, b) \to (x, y)$ is shorter than any path that shifts x at least two octaves in either direction *and* y at least two octaves in either direction.

Now we examine the cases where $m = -1, 0, 1$ *or* $n = -1, 0, 1$. First we will look at when $m = 0$.

- Let $m = 0$, then $n \geq 2$ or $n \leq -2$, so $12m^2 + 2m(x-a) + 12n^2 + 2n(y-b) = 12n^2 + 2n(y-b) \geq 0$ (from above).

  Similarly, in the case when $n = 0$ we have

- Let $n = 0$, then $m \geq 2$ or $m \leq -2$, so $12m^2 + 2m(x-a) + 12n^2 + 2n(y-b) = 12m^2 + 2m(x-a) \geq 0$ (from above).

  Again we have a contradiction, and have thus shown $(a,b) \to (x,y)$ is shorter than any path that shifts x or y at least two octaves and leaves the other in the starting octave.

  There remain 8 cases, two for each $m = -1, 1, n = -1, 1$.

  Let us start with $m = -1$.

- Let $m = -1$ then examine the distances for $(a,b) \to (x-12, y-12)$ and $(a,b) \to (x+12m, y+12n)$. Assume $d((a,b),(x-12,y-12)) > d((a,b),(x+12m,y+12n))$. So, we have

  $\sqrt{(a-x+12)^2 + (b-y+12)^2} > \sqrt{(a-x-12m)^2 + (b-y-12n)^2}$

  and can remove the square roots.

  $(a-x+12)^2 + (b-y+12)^2 > (a-x-12m)^2 + (b-y-12n)^2$

  Then we expand both sides of the inequality,

$$a^2 + x^2 + 12^2 - 2ax - 2(12x) + 2(12a) + b^2 + y^2 + 12^2 - 2by - 2(12y) + 2(12b) >$$
$$a^2+x^2+(12m)^2-2ax-2(12m)a+2(12m)x+b^2+y^2+(12n)^2-2by-2(12n)b+2(12n)y$$

  cancel matching terms,

$$12^2-2(12x)+2(12a)+12^2-2(12y)+2(12b) > (12m)^2-2(12m)a+2(12m)x+(12n)^2-2(12n)b+2(12n)y$$

  plug in $m = -1$,

$$12^2-2(12x)+2(12a)+12^2-2(12y)+2(12b) > 12^2+24a-24x+(12n)^2-2(12n)b+2(12n)y$$

  and cancel matching terms again.

$$12^2 - 24y + 24b > (12n)^2 - 24nb + 24ny$$

  Next we divide both sides by 12

$$12 - 2y + 2b > 12n^2 - 2nb + 2ny$$

  and move all terms to the right side.

$$0 > 12n^2 - 12 - 2nb - 2b + 2ny + 2y$$

  Then we factor the right side.

$$0 > 12(n^2-1) - 2b(n+1) + 2y(n+1)$$

  (1) Let $n \leq -2$, then we have

$$12(n^2-1) - 2b(n+1) + 2y(n+1)$$

  To make this the smallest it can be, let $b = 0$, $y = 12$,

$$\geq 12(n^2-1) - 2(0)(n+1) + 2(12)(n+1)$$

  which can be simplified.

$$= 12(n^2-1) + 24(n+1)$$

then we plug in $n = -2$.

$$= 12n^2 + 24n + 12 \geq 0\,\forall n$$

(2) Similarly, let $n \geq 2$, then we have

$$12(n^2 - 1) - 2b(n - 1) + 2y(n + 1)$$
$$\geq 12(n^2 - 1) - 2(12)(n - 1) + 2(0)(n + 1)$$
$$= 12(n^2 - 1) - 24(n - 1)$$
$$= 12n^2 - 24n + 12 \geq 0\,\forall n$$

Therefore, if $m = -1$, the path $(a, b) \to (x - 12, y - 12)$ is always shorter than shifting y by two octaves.

Next, take the cases when $m = 1$.

- Let $m = 1$ then examine the distances for $(a, b) \to (x + 12, y + 12)$ and $(a, b) \to (x+12m, y+12n)$. Assume $d((a, b), (x+12, y+12)) > d((a, b), (x+12m, y + 12n))$. So we have

$$\sqrt{(a - x - 12)^2 + (b - y - 12)^2} > \sqrt{(a - x - 12m)^2 + (b - y - 12n)^2}$$

and we remove the square roots,

$$(a - x - 12)^2 + (b - y - 12)^2 > (a - x - 12m)^2 + (b - y - 12n)^2$$

expand,

$$a^2 + x^2 + 12^2 - 2ax + 2(12x) - 2(12a) + b^2 + y^2 + 12^2 - 2by + 2(12y) - 2(12b) >$$
$$a^2+x^2+(12m)^2-2ax-2(12m)a+2(12m)x+b^2+y^2+(12n)^2-2by-2(12n)b+2(12n)y$$

cancel terms,

$$12^2+2(12x)-2(12a)+12^2+2(12y)-2(12b) > (12m)^2-2(12m)a+2(12m)x+(12n)^2-2(12n)b+2(12n)y$$

plug in $m = 1$,

$$12^2+2(12x)-2(12a)+12^2+2(12y)-2(12b) > 12^2-24a+24x+(12n)^2-2(12n)b+2(12n)y$$

and cancel terms again.

$$12^2 + 24y - 24b > (12n)^2 - 24nb + 24ny$$

Then we divide by 12

$$12 + 2y - 2b > 12n^2 - 2nb + 2ny$$

move all terms to right side

$$0 > 12n^2 - 12 - 2nb + 2b + 2ny - 2y$$

and factor.

$$0 > 12(n^2 - 1) - 2b(n - 1) + 2y(n - 1)$$

(1) Let $n \leq -2$, then we have

$$12(n^2 - 1) - 2b(n - 1) + 2y(n - 1)$$

So we make it smallest by letting $b = 0$ and $y = 12$,

$$\geq 12(n^2 - 1) - 2(0)(n - 1) + 2(12)(n - 1)$$

which can be simplified.

$$= 12(n^2 - 1) + 24(n - 1)$$

$$= 12n^2 + 24n + 12 \geq 0\,\forall n$$

(2) Similarly, let $n \geq 2$, then

$$\begin{aligned} &12(n^2-1) - 2b(n-1) + 2y(n-1) \\ &\geq 12(n^2-1) - 2(12)(n-1) + 2(0)(n-1) \\ &= 12(n^2-1) - 24(n-1) \\ &= 12n^2 - 24n + 12 \geq 0\,\forall n \end{aligned}$$

Therefore, if $m = 1$, the path $(a, b) \rightarrow (x + 12, y + 12)$ is always shorter than shifting y by two octaves.

The cases for $n = -1, 1$ follow similar arguments.

Therefore, shifting either pitch two octaves will never result in a minimum voice leading length. □

This narrows down our 18 path structures to exactly 18 paths where $m = n = 1$.

(1) $(a, b) \rightarrow (x, y)$
(2) $(a, b) \rightarrow (x - 12, y)$
(3) $(a, b) \rightarrow (x + 12, y)$
(4) $(a, b) \rightarrow (x, y - 12)$
(5) $(a, b) \rightarrow (x, y + 12)$
(6) $(a, b) \rightarrow (x - 12, y + 12)$
(7) $(a, b) \rightarrow (x + 12, y - 12)$
(8) $(a, b) \rightarrow (x + 12, y + 12)$
(9) $(a, b) \rightarrow (x - 12, y - 12)$
(10) $(a, b) \rightarrow (y, x)$
(11) $(a, b) \rightarrow (y - 12, x)$
(12) $(a, b) \rightarrow (y + 12, x)$
(13) $(a, b) \rightarrow (y, x - 12)$
(14) $(a, b) \rightarrow (y, x + 12)$
(15) $(a, b) \rightarrow (y - 12, x + 12)$
(16) $(a, b) \rightarrow (y + 12, x - 12)$
(17) $(a, b) \rightarrow (y + 12, x + 12)$
(18) $(a, b) \rightarrow (y - 12, x - 12)$

(See figure 15.)

We now systematically eliminate the paths by proving that one of the five from Theorem 3.4 will always be shorter than the rest.

Proof. (Theorem 3.4) We eliminate 13 of the 18 paths by comparing them individually with the five paths given in Theorem 3.4. In each case, we select one of the five paths and assume the distance is greater than the path we wish to eliminate. We then prove by contradiction that one of the five paths in the theorem is always shorter than each of the other 13 and therefore we do not need to consider those 13 when seeking a minimum distance function.

Recall the assumptions $a \leq b$ and $x \leq y$.

- Compare paths (2) and (11):
  Assume (11) > (2):

$$\sqrt{(a-y+12)^2 + (b-x)^2} > \sqrt{(a-x+12)^2 + (b-y)^2}$$

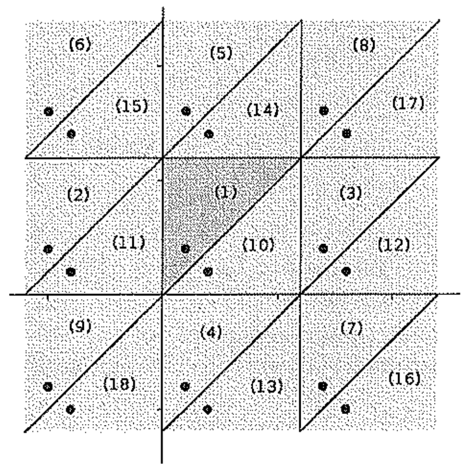


FIGURE 15. Each path takes the ending chord to a different tile. Here the tile with starting chords is colored blue and all of the tiles are labeled with the number of the path that takes a chord to that tile. Representations of an arbitrary chord have been plotted in each tile as well.

We remove the square roots and retain the inequality.

$$(a - y + 12)^2 + (b - x)^2 > (a - x + 12)^2 + (b - y)^2$$

Then we expand both sides of the inequality.

$$a^2+y^2+12^2-2ay+24a-24y+b^2-2bx+x^2 > a^2+x^2+12^2-2ax+24a-24x+b^2-2by+y^2$$

We eliminate matching terms.

$$-2ay - 24y - 2bx > -2ax - 24x - 2by$$

Then move the x terms to the right side and the y terms to the left side,

$$-2ay - 24y + 2by > -2ax - 24x + 2bx$$

and factor out x and y.

$$(-2a - 24 + 2b)y > (-2a - 24 + 2b)x$$

Since $(-2a - 24 + 2b) \neq 0$, we divide by $(-2a - 24 + 2b)$ and get

$$y < x$$

which is a contradiction, so $(11) \leq (2)$.

We proceed in a similar manner to elminate the remaining paths.

- Compare paths (3) and (14):
  Assume $(14) > (3)$:

$$\sqrt{(a - y)^2 + (b - x - 12)^2} > \sqrt{(a - x - 12)^2 + (b - y)^2}$$

$$(a - y)^2 + (b - x - 12)^2 > (a - x - 12)^2 + (b - y)^2$$

$$a^2-2ay+y^2+b^2+x^2+12^2-2bx-24b+24x > a^2+x^2+12^2-2ax-24a+24x+b^2-2by+y^2$$

$$-2ay - 2bx + 24b > -2ax - 24a - 2by$$

$$2by - 2bx + 24b > -2ax - 24a + 2ay$$

$$(-2x - 24 + 2y)b > (-2x - 24 + 2y)a$$
$$b < a$$

which is a contradiction, so (14) $\leq$ (3).

- Compare paths (4) and (11):
  Assume (11) > (4):

$$\sqrt{(a-y+12)^2 + (b-x)^2} > \sqrt{(a-x)^2 + (b-y+12)^2}$$
$$(a-y+12)^2 + (b-x)^2 > (a-x)^2 + (b-y+12)^2$$
$$a^2+y^2+12^2-2ay+24a-24y+b^2-2bx+x^2 > a^2-2ax+x^2+b^2+y^2+12^2-2by+24b-24y$$
$$-2ay + 24a - 2bx > -2ax - 2by + 24b$$
$$-2ay + 24a + 2ax > 2bx - 2by + 24b$$
$$(2x + 24 - 2y)a > (2x + 24 - 2y)b$$
$$a > b$$

which is a contradiction, so (11) $\leq$ (4).

- Compare paths (5) and (14):
  Assume (14) > (5):

$$\sqrt{(a-y)^2 + (b-x-12)^2} > \sqrt{(a-x)^2 + (b-y-12)^2}$$
$$(a-y)^2 + (b-x-12)^2 > (a-x)^2 + (b-y-12)^2$$
$$a^2-2ay+y^2+b^2+x^2+12^2-2bx-24b+24x > a^2-2ax+x^2+b^2+y^2+12^2-2by-24b+24y$$
$$-2ay - 2bx + 24x > -2ax - 2by + 24y$$
$$2ax - 2bx + 24x > 2ay - 2by + 24y$$
$$(2a + 24 - 2b)x > (2a + 24 - 2b)y$$
$$y < x$$

which is a contradiction, so (14) $\leq$ (5).

- Compare paths (6) and (1):
  Assume (1) > (6):

$$\sqrt{(a-x)^2 + (b-y)^2} > \sqrt{(a-x+12)^2 + (b-y-12)^2}$$
$$(a-y)^2 + (b-y)^2 > (a-x+12)^2 + (b-y-12)^2$$
$$a^2-2ax+x^2+b^2+-2by+y^2 > a^2+x^2+12^2-2ax-24x+24a+b^2+y^2+12^2-2by-24b+24y$$
$$0 > 12^2 - 24x + 24a + 12^2 - 24b + 24y$$
$$0 > 12 + (y-x) + (a-b)$$

Notice $-12 < a - b \leq 0$, so

$$0 > 12 + (y-x) + (a-b) > 12 + (y-x) - 12$$
$$0 > y - x$$
$$x > y$$

which is a contradiction, so (1) $\leq$ (6).

- Compare paths (7) and (1):
  Assume (1) > (7):

$$\sqrt{(a-x)^2+(b-y)^2} > \sqrt{(a-x-12)^2+(b-y+12)^2}$$

$$(a-y)^2+(b-y)^2 > (a-x-12)^2+(b-y+12)^2$$

$$a^2-2ax+x^2+b^2+-2by+y^2 > a^2+x^2+12^2-2ax+24x-24a+b^2+y^2+12^2-2by+24b-24y$$

$$0 > 12^2+24x-24a+12^2+24b-24y$$

$$0 > 12+(x-y)+(b-a)$$

  Notice $-12 < x-y \leq 0$, so

$$0 > 12+(x-y)+(b-a) > 12-12+(b-a)$$

$$0 > b-a$$

$$a > b$$

  which is a contradiction, so (1) $\leq$ (7).
- Compare paths (10) and (1):
  Assume (1) > (10):

$$\sqrt{(a-x)^2+(b-y)^2} > \sqrt{(a-y)^2+(b-x)^2}$$

$$(a-y)^2+(b-y)^2 > (a-y)^2+(b-x)^2$$

$$a^2-2ax+x^2+b^2+-2by+y^2 > a^2-2ay+y^2+b^2+-2bx+x^2$$

$$-2ax-2by > -2ay-2bx$$

$$-ax-by > -ay-bx$$

$$ay-by > ax-bx$$

$$(a-b)y > (a-b)x$$

$$y < x$$

  which is a contradiction, so (1) $\leq$ (10).
- Compare paths (12) and (14):
  Assume (14) > (12):

$$\sqrt{(a-y)^2+(b-x-12)^2} > \sqrt{(a-y-12)^2+(b-x)^2}$$

$$(a-y)^2+(b-x-12)^2 > (a-y-12)^2+(b-x)^2$$

$$a^2-2ay+y^2+b^2+x^2+12^2-2bx-24b+24x > a^2+y^2+12^2-2ay-24a+24y+b^2-2bx+x^2$$

$$-24b+24x > -24a+24y$$

$$x-b > y-a$$

$$a-b > y-x$$

  But $a-b \leq 0$ and $y-x \geq 0$
  which is a contradiction, so (14) $\leq$ (12).

- Compare paths (13) and (11):
  Assume (11) > (13):

$$\sqrt{(a-y+12)^2+(b-x)^2} > \sqrt{(a-y)^2+(b-x+12)^2}$$
$$(a-y+12)^2+(b-x)^2 > (a-y)^2+(b-x+12)^2$$
$$a^2+y^2+12^2-2ay+24a-24y+b^2-2bx+x^2 > a^2-2ay+y^2+b^2+x^2+12^2-2bx+24b-24x$$
$$24a-24y > 24b-24x$$
$$a-y > b-x$$
$$a-b > y-x$$

  But $a-b \leq 0$ and $y-x \geq 0$
  which is a contradiction, so (11) $\leq$ (13).
- Compare paths (15) and (1):
  Assume (1) > (15):

$$\sqrt{(a-x)^2+(b-y)^2} > \sqrt{(a-y+12)^2+(b-x-12)^2}$$
$$(a-y)^2+(b-y)^2 > (a-y+12)^2+(b-x-12)^2$$
$$a^2-2ax+x^2+b^2+-2by+y^2 > a^2+y^2+12^2-2ay+24a-24y+b^2+y^2+12^2-2bx+24x-24b$$
$$-2ax-2by > 12^2-2ay+24a-24y+12^2-2bx+24x-24b$$
$$0 > 12^2+12a-12b-12y-ay+by+12x+ax-bx$$
$$0 > 12(12+a-b)-y(12+a-b)+x(12+a-b)$$
$$0 > 12+(x-y)$$

  But $-12 < x-y \leq 0$
  which is a contradiction, so (1) $\leq$ (15).
- Compare paths (16) and (1):
  Assume (1) > (16):

$$\sqrt{(a-x)^2+(b-y)^2} > \sqrt{(a-y-12)^2+(b-x+12)^2}$$
$$(a-y)^2+(b-y)^2 > (a-y-12)^2+(b-x+12)^2$$
$$a^2-2ax+x^2+b^2+-2by+y^2 > a^2+y^2+12^2-2ay-24a+24y+b^2+y^2+12^2-2bx-24x+24b$$
$$-2ax-2by > 12^2-2ay-24a+24y+12^2-2bx-24x+24b$$
$$0 > 12^2-12a+12b+12y-ay+by-12x+ax-bx$$
$$0 > 12(12-a+b)+y(12-a+b)-x(12-a+b)$$
$$0 > 12+(y-x)$$

  But $0 \leq y-x < 12$
  which is a contradiction, so (1) $\leq$ (16).
- Compare paths (17) and (8):
  Assume (8) > (17):

$$\sqrt{(a-x-12)^2+(b-y-12)^2} > \sqrt{(a-y-12)^2+(b-x-12)^2}$$
$$(a-x-12)^2+(b-y-12)^2 > (a-y-12)^2+(b-x-12)^2$$

$$a^2+x^2+12^2-2ax-24a+24x+b^2+y^2+12^2-2by-24b+24y$$
$$> a^2+y^2+12^2-2ay-24a+24y+b^2+x^2+12^2-2bx-24b+24x$$
$$-2ax-2by > -2ay-2bx$$
$$ay-by > ax-bx$$

$$(a-b)y > (a-b)x$$
$$y < x$$

which is a contradiction, so (8) $\leq$ (17).

- Compare paths (18) and (9):
  Assume (9) $>$ (18):

$$\sqrt{(a-x+12)^2+(b-y+12)^2} > \sqrt{(a-y+12)^2+(b-x+12)^2}$$
$$(a-x+12)^2+(b-y+12)^2 > (a-y-12)^2+(b-x-12)^2$$
$$a^2+x^2+12^2-2ax+24a-24x+b^2+y^2+12^2-2by+24b-24y$$
$$> a^2+y^2+12^2-2ay+24a-24y+b^2+x^2+12^2-2bx+24b-24x$$
$$-2ax-2by > -2ay-2bx$$
$$ay-by > ax-bx$$
$$(a-b)y > (a-b)x$$
$$y < x$$

which is a contradiction, so (9) $\leq$ (18).

□

This theorem makes it much simpler to find the distance between two chords: there are only five possible distance functions to check.

Corollary 3.1. *First, let $\bar{d}((a,b),(x,y))$ be the Euclidean distance defined for pitch class space.*

Then we define distance in chord space as

$$d(\{a,b\},\{x,y\}) = \min\{\bar{d}((a,b),(x,y)),$$
$$\bar{d}((a,b),(x-12,y-12)),$$
$$\bar{d}((a,b),(x+12,y+12)),$$
$$\bar{d}((a,b),(y-12,x)),$$
$$\bar{d}((a,b),(y,x+12))\},$$

the minimum of the five paths in Theorem 3.4.

Proof. The proof follows directly from Theorem 3.4. Since we define distance to be the minimum distance to each pitch space representation of a chord, and since Theorem 3.4 tells us only five representatives can have the minimum distance, then the minimun of all paths equals the minimum of those five paths. □

As mentioned above, we also want to find paths that do not contain voice crossings. We see that all five of the potential paths for a shortest distance exist between the lines $y = x$ and $y = x + 12$, which correspond to voice crossings.

Therefore, it is clear that none of the minimal voice leadings intersect these lines, making it reasonable to suspect that our five minimal voice leading paths do not cross.

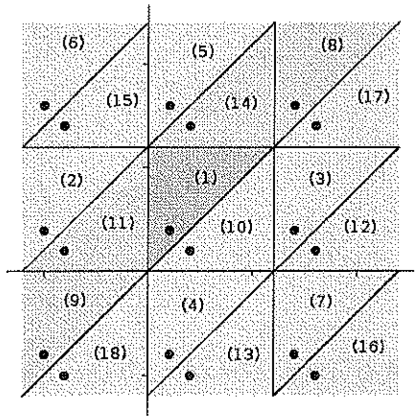
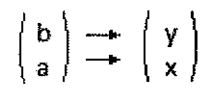


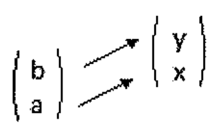
FIGURE 16. The five potential paths are highlighted.

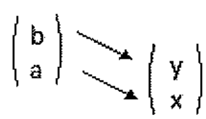
Corollary 3.2. *The shortest distance between two two-note chords does not contain voice crossings.*

Proof. The shortest distance is given by one of the five functions in Theorem 3.4. In the function $(a, b) \to (x, y)$, $a \to x$ and $b \to y$. Since $a \leq b$ and $x \leq y$, there is not a voice crossing.



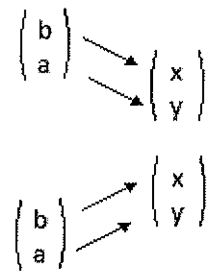
Similarly, when we shift x and y in the same direction, we preserve the noncrossed voice leading of $(a, b) \to (x, y)$ since $x \leq y$ implies $x + 12 \leq y + 12$ and $x \leq y$ implies $x - 12 \leq y - 12$.





When we switch the coordinates to $(a, b) \to (y, x)$, we create a voice crossing.

It is uncrossed when we move y down an octave or x up an octave because $a, b, x, y \in \mathbb{Z}_{12}$.



∎

These results show that our distance function satisfies a preference for conjunct melodic motion as well as uncrossed voice leadings.

3.5. **Proof of Metric.** We have referred to our function as defining a distance, but must prove that it actually does.

Theorem 3.5. *For chords x and y, the function $d(x,y)$ defines a metric.*

A metric satisfies the following four constraints for all x, y, z:

(1) $d(x,y) \geq 0$
(2) $d(x,y) = 0$ if and only if $x = y$
(3) $d(x,y) = d(y,x)$
(4) $d(x,z) \leq d(x,y) + d(y,z)$

Some of these traits are easily observed in our function due to it being a minimum of Euclidean distance functions.

Proof. (1) Since the Euclidean metric is non-negative for all x and y, the minimum of five Euclidean distance functions must also be non-negative.

(2) Since the Euclidean metric is zero if and only if $x = y$, the minimum of five Euclidean distance functions must also be zero if and only if $x = y$.

(3) Since the Euclidean metric has $d(x,y) = d(y,x)$, the value of each of the five individual Euclidean distance functions has $d(x,y) = d(y,x)$, and therefore the minimum of the five Euclidean distance functions will be the same for (x,y) and (y,x).

(4) Triangle Inequality:

We prove the following lemma to simplify the proof of the triangle inequality. The lemma states that when the starting chord is shifted to another octave, the minimal distance is the same and the ending chord representative is shifted in the same manner as the starting chord. This comes from the translational symmetry of pitch space: we can find another representative for the ending chord that gives us a congruent voice leading path.

Lemma 3.2. *Consider the arbitrary chord progression $\{a,b\} \to \{x,y\}$.*

Let $a \leq b$ for $a, b \in \mathbb{Z}_{12}$ and (x', y') is the minmizing representative of $\{x,y\}$, so we have $(a,b) \to (x',y')$ is a minimal voice leading.

Then, the distance for $(a + 12m, b + 12n) \to (x' + 12m, y' + 12n)$ with $m, n \in \mathbb{Z}$ is equal to the distance for $(a,b) \to (x',y')$.

Proof. Let $a \leq b$ and $x \leq y$ for $a, b, x, y \in \mathbb{Z}_{12}$.

Let (x', y') be the pitch set such that $(a, b) \rightarrow (x', y')$ is the minimal voice leading for the progression $\{a, b\} \rightarrow \{x, y\}$.

Then $(a + 12m, b + 12n)$ for $m, n \in \mathbb{Z}$ represents all chord equivalent pitch sets for (a, b).

And, $(x' + 12m, y' + 12n)$ for $m, n \in \mathbb{Z}$ represents all chord equivalent pitch sets for (x', y').

Then we want to show $(a+12m, b+12n) \rightarrow (x'+12m, y'+12n)$ is equal to $(a, b) \rightarrow (x', y')$.

Examine:

$$\begin{aligned} &\bar{d}((a + 12m, b + 12n), (x' + 12m, y' + 12n)) \\ &= \sqrt{((a + 12m) - (x' + 12m))^2 + ((b + 12n) - (y' + 12n))^2} \\ &= \sqrt{(a + 12m - x' - 12m)^2 + (b + 12n - y' - 12n)^2} \\ &= \sqrt{(a - x')^2 + (b - y')^2} \\ &= \bar{d}((a, b), (x, y)) \end{aligned}$$

which is the distance for $(a, b) \rightarrow (x', y')$.

□

We return to the proof of the Triangle Inequality: Our goal is to show $d(x, z) \leq d(x, y) + d(y, z)$ for our function. First we will define these distances, then restate our goal.

- Consider $d(x, y)$.
  Since x is a starting chord, we know the representative of x we use has $a \leq b$ for $a, b \in \mathbb{Z}_{12}$, so let:
  $$x = (a_1, b_1)$$
  Since the ending chord y could be represented by a pitch set from any of the five voice leadings, let us call the minimal picth set representative y'.
  $$y' = (a'_2, b'_2)$$
  This distinguishes it from the pitch sets whose location we know.
- Similarly, consider $d(y, z)$.
  Since y is a starting chord, it is not necessarily represented by the same pitch set as y'. It is now fixed in the same octave as x, so let:
  $$y = (a_2, b_2)$$
  Let the ending pitch set be z'
  $$z' = (a'_3, b'_3)$$
- Finally, consider $d(x, z)$.
  The starting chord, x, is the same pitch set as above,
  $$x = (a_1, b_1)$$
  Let the ending pitch set be z'', because it is not necessarily the same representation as z', and is not necessarily in the same octave as the starting representatives of x and y.
  $$z'' = (a''_3, b''_3)$$

- Our goal now is to show: $\bar{d}(x,z'') \leq \bar{d}(x,y') + \bar{d}(y,z')$
  We have:

$$\bar{d}(x,y') + \bar{d}(y,z') = \bar{d}(x,y') + \bar{d}(y',z''')$$

  for some z''' by Lemma 3.2.

$$\bar{d}(x,y') + \bar{d}(y',z''') \geq \bar{d}(x,z''')$$

  since we are using the Euclidean metric.

$$\bar{d}(x,z''') \geq \bar{d}(x,z'')$$

  because we defined z'' to be the representation of the chord z that produces the minimal distance. Therefore, $\bar{d}(x,z'') \leq \bar{d}(x,y') + \bar{d}(y,z')$. ☐

4. Three-Note Chords

We now extend our results to three-note chords. The set-up is similar, but now instead of being in Euclidean 2-space, pitch sets are in Euclidean 3-space. A "tile" in pitch class space is now a "box," and chord space is now a torus rather than a Mobius strip.

4.1. **Geometric Spaces.** In three dimensions, we now use $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ for pitch space. Voice crossings occur along the planes $x = y$, $y = z$, and $x = z$.

Theorem 4.1. *Pitch space is equivalent to* $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

Proof. The proof is identical to that of two note pitch space.

We write a one-to-one correspondence from pitch space to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ with the same approach as before. ☐

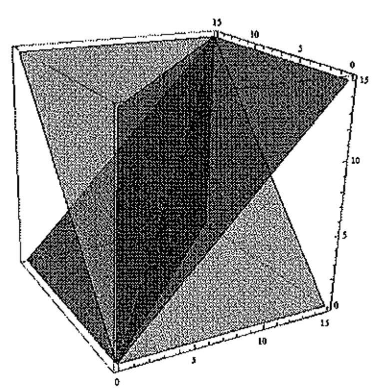


FIGURE 17. The planes $x = y$, $y = z$, and $x = z$ divide pitch space into six segments.

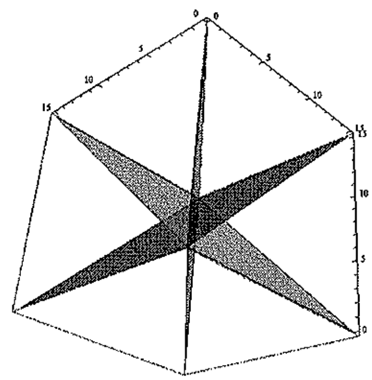


FIGURE 18. We see figure 17 more clearly divides the region by looking up from the origin.

Next we abstract to pitch class space. Pitch class space is the $12 \times 12 \times 12$ box since pitch classes are mod12. The same concepts of two-note pitch class space apply to three-note pitch class space. We see that the box contains six copies of each chord since there are six permutations on a set of three.

Theorem 4.2. *Pitch class space is equivalent to* $\mathbb{Z}_{12} \times \mathbb{Z}_{12} \times \mathbb{Z}_{12}$.

Proof. The proof is identical to that of two note pitch class space.

We write a one-to-one correspondence from pitch class space to $\mathbb{Z}_{12} \times \mathbb{Z}_{12} \times \mathbb{Z}_{12}$ with the same approach as before. □

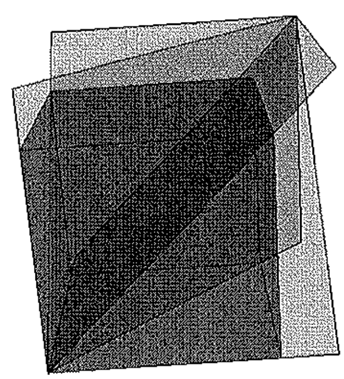


FIGURE 19. The six pitch class representations of $\{1, 2, 3\}$ and $\{4, 6, 8\}$ are plotted in pitch class space.

The six copies are separated by the unison planes. Just as before with two-note chords, we want only one copy of each chord to be in chord space. Here, we manipulate a single slice of pitch class space to form the torus for chord space. We divide this slice along two planes that partition it into equal segments. Then we flip and rotate those pieces and put them back together to get the torus.

Theorem 4.3. *Chord space is equivalent to the torus with a triangular cross section.*

Proof. We can create a one-to-one correspondence between the two spaces.

We write a correspondence, $\{p_1, p_2, p_3\} \to \mathbb{Z}_{12} \times \mathbb{Z}_{12} \times \mathbb{Z}_{12}$. We see $\mathbb{Z}_{12} \times \mathbb{Z}_{12} \times \mathbb{Z}_{12}$ is topologically equivalent to the 3-torus, $\mathbb{T}^3$, by identifying opposite faces.

But, this correspondence is not one-to-one; it is one-to-six because each chord is mapped to six ordered pairs.

To make it one-to-one we mod out by permutations on a set of three, or S_3. The simplest way to do this is to take one segment of the box diveded by the planes in figure 19. We then rearrange it into a torus (see figure 20). The torus puts chords close to each other that we would expect to be close by wrapping around the edges. □

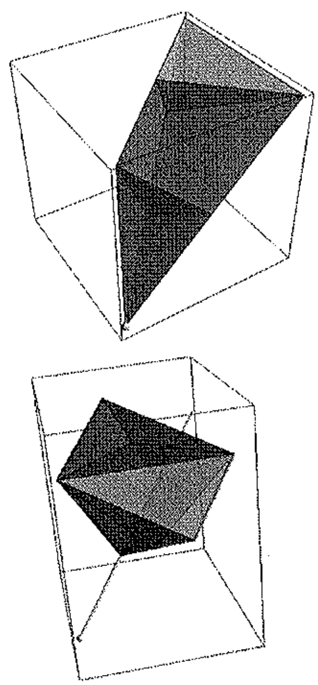


FIGURE 20. This shows the transformation from a single slice of pitch class space to the chord space torus.

4.2. **Behavior in Chord Space.** The representation of three-note chord space is a torus with an equilateral triangle cross section and a 120 degree twist. Similar to two-space, along the edge are the unison chords, in which all three pitches are of the same pitch class, $\{C, C, C\}$ for example. Along the faces, two of the pitches are of the same class, $\{C, C, D\}$ for example. Also similar to two-space, the center of the torus contains sets that divide the octave evenly. In two-space, we see tritones along the center, which are 6 semitones apart like $\{0,6\}$ or $\{C, F^{\#}\}$; in three-space

4 semitones divides the octave, so the set {0,4,8} or $\{C, E, G^{\#}\}$ is situated along the center.

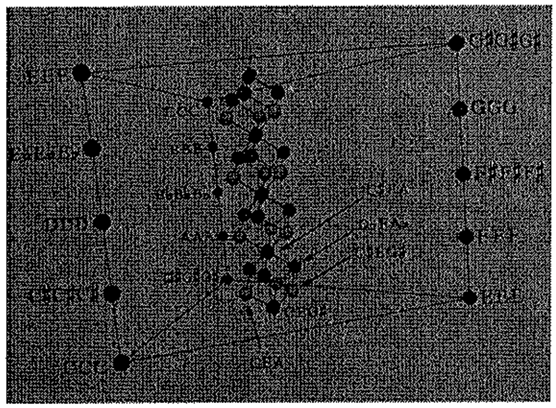


FIGURE 21. Tymoczko's representation of the torus [4].

When examining voice leadings in the torus, the triangular faces are connected with a rotation of 120 degrees, so by passing off those edges, one just recenters from the other end. The face, which contains unison pitches, corresponds to voice crossings. When a voice leading contains a crossing, it appears to bounce off the face of the torus.

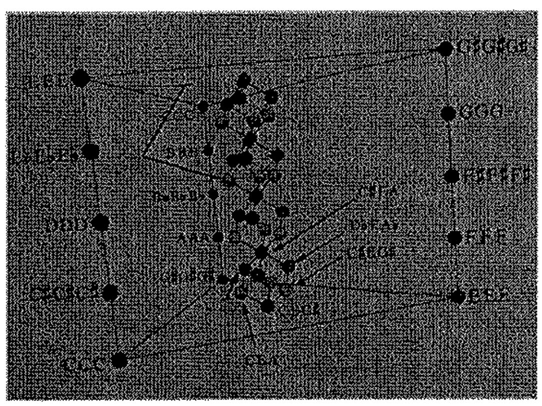


FIGURE 22. A voice leading in the torus bouces off the back face and passes through the top and bottom faces [4].

We define distance in the same manner as before.

Definition. The distance between two three-note chords is the minimum of all possible paths between them.

Again, each path in chord space corresponds to a path between different pairs of pitch sets. We use the same technique of pulling the chords back into pitch space and examining the paths there in order to find a distance function for chords.

4.3. **Measuring Chord Distances.** For three-note chords, we examine the voice leadings from $\{a, b, c\} \rightarrow \{x, y, z\}$. We will again restrict the starting chord to exist in a single slice of pitch class space. Thus, we let $a \leq b \leq c$ for $a, b, c \in \mathbb{Z}_{12}$.

We construct the potential paths as we did for two-note chords and likewise restrict them to single octave shifts in each voice. Instead of 18 paths, there are 27 (cubes) $\times$ 6 (cells) = 162 paths.

4.4. **Minimal Voice Leading.** To narrow the paths from 162, we use Mathematica. Our program checks the distance from each (starting) chord to each other (ending) chord, and records which functions are used to produce a minimum. Thus, we have the following theorem.

Theorem 4.4. *There exist twenty seven paths in pitch space that result in a minimal distance between three-note chords:*
For $x \leq y \leq z$:

$$(a,b,c) \rightarrow (x,y,z)$$
$$(a,b,c) \rightarrow (x,y,z+12)$$
$$(a,b,c) \rightarrow (x,y+12,z+12)$$
$$(a,b,c) \rightarrow (x+12,y+12,z+12)$$
$$(a,b,c) \rightarrow (x-12,y,z)$$
$$(a,b,c) \rightarrow (x-12,y-12,z)$$
$$(a,b,c) \rightarrow (x-12,y-12,z-12)$$
$$(a,b,c) \rightarrow (y-12,x,z)$$
$$(a,b,c) \rightarrow (y,x+12,z+12)$$
$$(a,b,c) \rightarrow (y-12,x,z+12)$$
$$(a,b,c) \rightarrow (y-12,x+12,z+12)$$
$$(a,b,c) \rightarrow (z-12,y,x+12)$$
$$(a,b,c) \rightarrow (x,z,y+12)$$
$$(a,b,c) \rightarrow (x-12,z-12,y)$$
$$(a,b,c) \rightarrow (x-12,z,y+12)$$
$$(a,b,c) \rightarrow (x-12,z-12,y+12)$$
$$(a,b,c) \rightarrow (y,z,x+12)$$
$$(a,b,c) \rightarrow (y-12,z-12,x)$$
$$(a,b,c) \rightarrow (y-12,z,x+12)$$
$$(a,b,c) \rightarrow (y-12,z-12,x+12)$$
$$(a,b,c) \rightarrow (z-12,x,y)$$
$$(a,b,c) \rightarrow (z,x+12,y+12)$$
$$(a,b,c) \rightarrow (z-12,x,y+12)$$
$$(a,b,c) \rightarrow (z-12,x+12,y+12)$$
$$(a,b,c) \rightarrow (x-12,y,z+12)$$
$$(a,b,c) \rightarrow (x-12,y+12,z+12)$$
$$(a,b,c) \rightarrow (x-12,y-12,z+12)$$

These paths each end in a different segment of pitch space, so we can visualize all of the possible segments that get mapped to. This is comparable to the five highlighted regions in figure 16. We see that none of the segments exist outside of the planes that correspond to voice crossings. This is similar to the two note figure, in which all possible ending locations of the minimal voice leading occurred between the lines $y = x$ and $y = x + 12$.

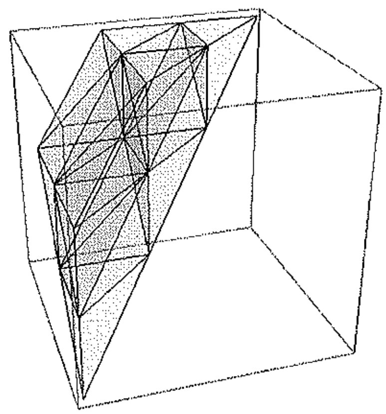


FIGURE 23. The starting pitch sets exist in the yellow segment. Each other segment contains a representation of a chord that is nearest some pitch set in the starting segment.

Another way to visualize the location of the ending chords is with points. We plot the ending pitch set representation where it gets sent to by one of the 27 possible minimal functions.

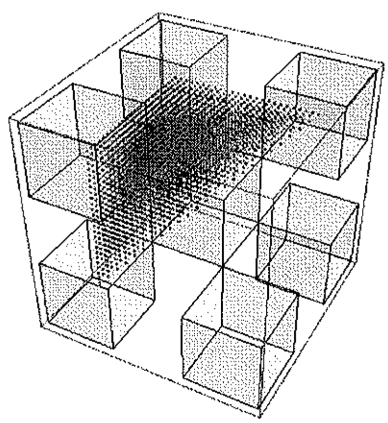


FIGURE 24. Each ending chord is plotted where one of the minimizing functions maps it. The starting tile is in the center with a few other nearby tiles around it for perspective. Each ending pitch set is color coded by the function that is used.

This image is a composite of each of the 27 individual functions. Using Mathematica, we can also examine them each individually. Each function maps the chords in one direction, and when that function produces a minimal distance, it places a dot at the location it mapped to. This makes it appear as clusters of points around the starting octave.

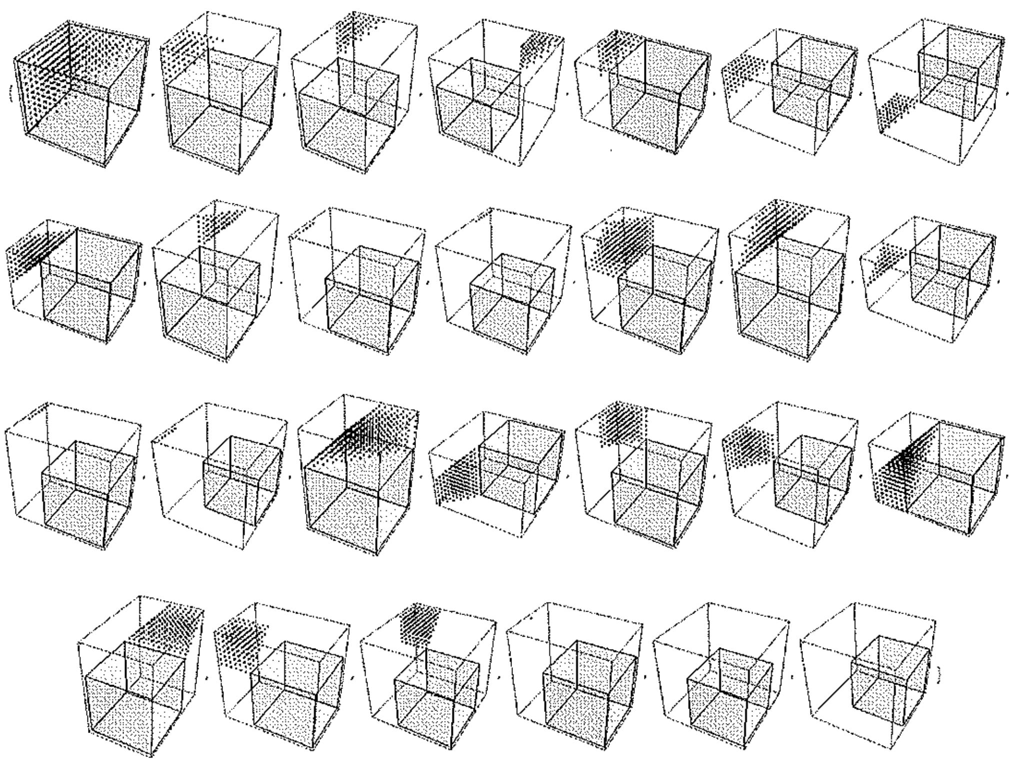


FIGURE 25. Each individual function finds minimum distances by sending the chord representations to the regions made visible by the points.

We create a similar set of images by plotting the voice leadings that each of the 27 functions finds as a minimum. This means we connect the starting chord and the ending chord with a line, rather than just plotting the ending chord.

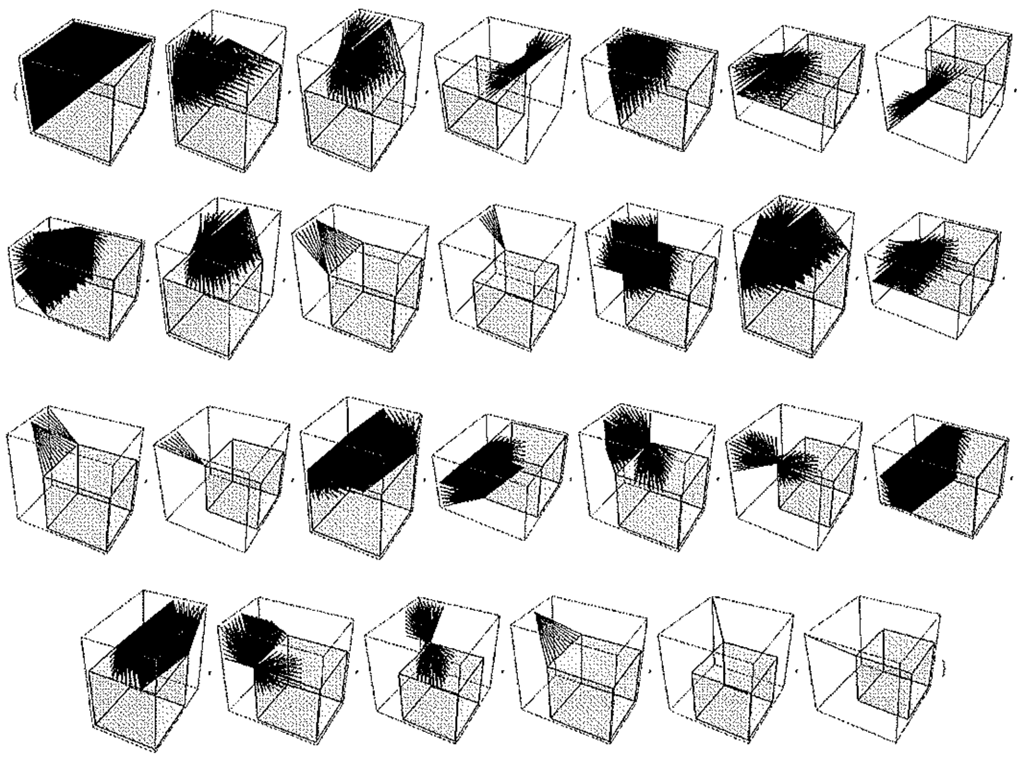


FIGURE 26. Each voice leading is plotted, starting within the cube - specifically in the segement with $x \leq y \leq z$ - and ending in the same regions as figure 25

These images are useful because they reveal the location of the starting chords, so we see where chords are clustered when each function is used. We also get an idea of how often each function is used to produce a minimum. For instance, some of the functions have one point/line, meaning there is only one instance when that function produces a minimum. We can also see a certain amount of symmetry between the functions when we plot them this way.

All of the figures depict the location of the nearest ending chord for each starting chord. The Mathematica programs take each starting chord and find the minimal paths, then plot them. Thus, we see that none of the chords go beyond a single octave. We also see that the 27 paths all send the chords along uncrossed voice leadings. Thus the results of our two note analysis still hold.

5. Future Work

Work with this function can continue in several areas. Since we have a metric, there is potential for topological analysis of the spaces. We define a metric space, (X, d), where X is the set of chords and d is the distance function. With this, we can induce the metric topology on the space.

Another approach is to gather data on existing compositions. We quantify the voice leading relationships and can analyze trends in different eras of music. Tymoczko's claim applies to music of other cultures as well. Our analysis could be easily translated to work with foreign tuning systems that divide the octave differently, and therefore would not have integer values when plotted in the spaces we have been using.

Perhaps the most interesting continuation would be to incorporate acoustic distances and have a computer program write musical compositions. This would provide another way to evaluate our function and the validity of Tymoczko's claim. A composition would allow an auditory analysis of the usefulness of the function.

Some of the challenges with continuing work deal with cardinality. One would have to define a way to deal with cardinality changes, such as a three note chord moving to a four note chord. Also, beyond three dimensions, we cannot easily visualize the spaces. Each voice corresponds to another dimension, so an orchestra would have many dimensions compared to a trio or duet. Lastly, it is common for chords to be broken up in compositions. Further work may need to consider how to handle this situation.

6. Conclusion

We have thus constructed a method for measuring voice leading sizes in chord progressions. We were able to minimize this size in order to comply with the component of tonality Tymoczko calls "conjunct melodic motion." We also showed that the minimal paths our function proposes do not contain voice crossings. The results give us one way to observe nearby chords. Being able to quantify the relationships of chords offers an alternate approach to certain areas of music theory and analysis.

7. Appendix A

Mathematica Code

```
F[1][a_,b_,c_,x_,y_,z_]:=Sqrt[(x-a)^2 + (y - b)^2 + (z - c)^2];
F[2][a_,b_,c_,x_,y_,z_]:=Sqrt[(x-a)^2 + (y - b)^2 + (z + 12 - c)^2];
F[3][a_,b_,c_,x_,y_,z_]:=Sqrt[(x-a)^2 + (y + 12 - b)^2 + (z + 12 - c)^2];
F[4][a_,b_,c_,x_,y_,z_]:=Sqrt[(x+12-a)^2 + (y + 12 - b)^2 + (z + 12 - c)^2];
F[5][a_,b_,c_,x_,y_,z_]:=Sqrt[(x-12-a)^2 + (y - b)^2 + (z - c)^2];
F[6][a_,b_,c_,x_,y_,z_]:=Sqrt[(x-12-a)^2 + (y - 12 - b)^2 + (z - c)^2];
F[7][a_,b_,c_,x_,y_,z_]:=Sqrt[(x-12-a)^2 + (y - 12 - b)^2 + (z - 12 - c)^2];
F[8][a_,b_,c_,x_,y_,z_]:=Sqrt[(y-12-a)^2 + (x - b)^2 + (z - c)^2];
F[9][a_,b_,c_,x_,y_,z_]:=Sqrt[(y-a)^2 + (x + 12 - b)^2 + (z + 12 - c)^2];
F[10][a_,b_,c_,x_,y_,z_]:=Sqrt[(y-12-a)^2 + (x - b)^2 + (z + 12 - c)^2];
F[11][a_,b_,c_,x_,y_,z_]:=Sqrt[(y-12-a)^2 + (x + 12 - b)^2 + (z + 12 - c)^2];
F[12][a_,b_,c_,x_,y_,z_]:=Sqrt[(z-12-a)^2 + (y - b)^2 + (x + 12 - c)^2];
F[13][a_,b_,c_,x_,y_,z_]:=Sqrt[(x-a)^2 + (z - b)^2 + (y + 12 - c)^2];
F[14][a_,b_,c_,x_,y_,z_]:=Sqrt[(x-12-a)^2 + (z - 12 - b)^2 + (y - c)^2];
F[15][a_,b_,c_,x_,y_,z_]:=Sqrt[(x-12-a)^2 + (z - b)^2 + (y + 12 - c)^2];
F[16][a_,b_,c_,x_,y_,z_]:=Sqrt[(x-12-a)^2 + (z - 12 - b)^2 + (y + 12 - c)^2];
F[17][a_,b_,c_,x_,y_,z_]:=Sqrt[(y-a)^2 + (z - b)^2 + (x + 12 - c)^2];
F[18][a_,b_,c_,x_,y_,z_]:=Sqrt[(y-12-a)^2 + (z - 12 - b)^2 + (x - c)^2];
F[19][a_,b_,c_,x_,y_,z_]:=Sqrt[(y-12-a)^2 + (z - b)^2 + (x + 12 - c)^2];
F[20][a_,b_,c_,x_,y_,z_]:=Sqrt[(y-12-a)^2 + (z - 12 - b)^2 + (x + 12 - c)^2];
```

```
F[21][a_,b_,c_,x_,y_,z_]:=Sqrt[(z-12-a)^2 + (x - b)^2 + (y - c)^2];
F[22][a_,b_,c_,x_,y_,z_]:=Sqrt[(z-a)^2 + (x + 12 - b)^2 + (y + 12 - c)^2];
F[23][a_,b_,c_,x_,y_,z_]:=Sqrt[(z-12-a)^2 + (x - b)^2 + (y + 12 - c)^2];
F[24][a_,b_,c_,x_,y_,z_]:=Sqrt[(z-12-a)^2 + (x + 12 - b)^2 + (y + 12 - c)^2];
F[25][a_,b_,c_,x_,y_,z_]:=Sqrt[(x-12-a)^2 + (y - b)^2 + (z + 12 - c)^2];
F[26][a_,b_,c_,x_,y_,z_]:=Sqrt[(x-12-a)^2 + (y + 12 - b)^2 + (z + 12 - c)^2];
F[27][a_,b_,c_,x_,y_,z_]:=Sqrt[(x-12-a)^2 + (y - 12 - b)^2 + (z + 12 - c)^2];
(*F[28][a_,b_,c_,x_,y_,z_]:=Sqrt[(z-a)^2 + (y + 12 - b)^2 + (x + 24 - c)^2];
F[29][a_,b_,c_,x_,y_,z_]:=Sqrt[(z-24-a)^2 + (y - 12 - b)^2 + (x - c)^2];*)
x=Range[0,12];
y=Range[0,12];
z=Range[0,12];
a=Range[0,12];
b=Range[0,12];
c=Range[0,12];
Do[fList[r]={},{r,1,27}]
funcs={}
Do[
If[a[[l]]≤ b[[n]] ≤ c[[s]],
Do[
If[x[[i]]≤ y[[j]] ≤ z[[k]],
min=100;minFs={};
Do[
m=F[r][a[[l]],b[[n]],c[[s]],x[[i]],y[[j]],z[[k]]];If[m≤ min, min = m];,
{r,1,27}];
Do[
w=F[r][a[[l]],b[[n]],c[[s]],x[[i]],y[[j]],z[[k]]];If[w==min,AppendTo[minFs,F[r]];AppendTo[fList[r],{{a[[l]],b[[n]],c[[s
{r,1,27}];
Do[If[!MemberQ[funcs,minFs[[q]]],AppendTo[funcs,minFs[[q]]];Print[funcs,"[",a[[l]],",",b[[n]],",",c[[s]],"]","->
{i,1,Length[x]},{j,1,Length[y]},{k,1,Length[z]}]];,
{l,1,Length[a]},{n,1,Length[b]},{s,1,Length[c]}]
Do[newList[i]={},{i,1,27}]
Do[
Do[
AppendTo[newList[1],{{fList[1][[i,1,1]],fList[1][[i,1,2]],fList[1][[i,1,3]]},{fList[1][[i,2,1]],fList[1][[i,2,2]],fList[1][[i,2
Do[
AppendTo[newList[2],{{fList[2][[i,1,1]],fList[2][[i,1,2]],fList[2][[i,1,3]]},{fList[2][[i,2,1]],fList[2][[i,2,2]],fList[2][[i,2
Do[AppendTo[newList[3],{{fList[3][[i,1,1]],fList[3][[i,1,2]],fList[3][[i,1,3]]},{fList[3][[i,2,1]],fList[3][[i,2,2]]+12,fLi
Do[
AppendTo[newList[4],{{fList[4][[i,1,1]],fList[4][[i,1,2]],fList[4][[i,1,3]]},{fList[4][[i,2,1]]+12,fList[4][[i,2,2]]+12,fL
Do[
AppendTo[newList[5],{{fList[5][[i,1,1]],fList[5][[i,1,2]],fList[5][[i,1,3]]},{fList[5][[i,2,1]]-
12,fList[5][[i,2,2]],fList[5][[i,2,3]]}}];,{i,1,Length[fList[5]]}];
Do[
AppendTo[newList[6],{{fList[6][[i,1,1]],fList[6][[i,1,2]],fList[6][[i,1,3]]},{fList[6][[i,2,1]]-
12,fList[6][[i,2,2]]-12,fList[6][[i,2,3]]}}];,{i,1,Length[fList[6]]}];
Do[
```

```
AppendTo[newList[7],{{fList[7][[i,1,1]],fList[7][[i,1,2]],fList[7][[i,1,3]]},{fList[7][[i,2,1]]-
12,fList[7][[i,2,2]]-12,fList[7][[i,2,3]]-12}}];,{i,1,Length[fList[7]]}];
Do[
AppendTo[newList[8],{{fList[8][[i,1,1]],fList[8][[i,1,2]],fList[8][[i,1,3]]},{fList[8][[i,2,2]]-
12,fList[8][[i,2,1]],fList[8][[i,2,3]]}}];,{i,1,Length[fList[8]]}];
Do[
AppendTo[newList[9],{{fList[9][[i,1,1]],fList[9][[i,1,2]],fList[9][[i,1,3]]},{fList[9][[i,2,2]],fList[9][[i,2,1]]+12,fList[9
Do[
AppendTo[newList[10],{{fList[10][[i,1,1]],fList[10][[i,1,2]],fList[10][[i,1,3]]},{fList[10][[i,2,2]]-
12,fList[10][[i,2,1]],fList[10][[i,2,3]]+12}}];,{i,1,Length[fList[10]]}];
Do[
AppendTo[newList[11],{{fList[11][[i,1,1]],fList[11][[i,1,2]],fList[11][[i,1,3]]},{fList[11][[i,2,2]]-
12,fList[11][[i,2,1]]+12,fList[11][[i,2,3]]+12}}];,{i,1,Length[fList[11]]}];
Do[
AppendTo[newList[12],{{fList[12][[i,1,1]],fList[12][[i,1,2]],fList[12][[i,1,3]]},{fList[12][[i,2,3]]-
12,fList[12][[i,2,2]],fList[12][[i,2,1]]+12}}];,{i,1,Length[fList[12]]}];
Do[
AppendTo[newList[13],{{fList[13][[i,1,1]],fList[13][[i,1,2]],fList[13][[i,1,3]]},{fList[13][[i,2,1]],fList[13][[i,2,3]],fLis
Do[
AppendTo[newList[14],{{fList[14][[i,1,1]],fList[14][[i,1,2]],fList[14][[i,1,3]]},{fList[14][[i,2,1]]-
12,fList[14][[i,2,3]]-12,fList[14][[i,2,2]]}}];,{i,1,Length[fList[14]]}];
Do[
AppendTo[newList[15],{{fList[15][[i,1,1]],fList[15][[i,1,2]],fList[15][[i,1,3]]},{fList[15][[i,2,1]]-
12,fList[15][[i,2,3]],fList[15][[i,2,2]]+12}}];,{i,1,Length[fList[15]]}];
Do[
AppendTo[newList[16],{{fList[16][[i,1,1]],fList[16][[i,1,2]],fList[16][[i,1,3]]},{fList[16][[i,2,1]]-
12,fList[16][[i,2,3]]-12,fList[16][[i,2,2]]+12}}];,{i,1,Length[fList[16]]}];
Do[
AppendTo[newList[17],{{fList[17][[i,1,1]],fList[17][[i,1,2]],fList[17][[i,1,3]]},{fList[17][[i,2,2]],fList[17][[i,2,3]],fLis
Do[
AppendTo[newList[18],{{fList[18][[i,1,1]],fList[18][[i,1,2]],fList[18][[i,1,3]]},{fList[18][[i,2,2]]-
12,fList[18][[i,2,3]]-12,fList[18][[i,2,1]]}}];,{i,1,Length[fList[18]]}];
Do[
AppendTo[newList[19],{{fList[19][[i,1,1]],fList[19][[i,1,2]],fList[19][[i,1,3]]},{fList[19][[i,2,2]]-
12,fList[19][[i,2,3]],fList[19][[i,2,1]]+12}}];,{i,1,Length[fList[19]]}];
Do[
AppendTo[newList[20],{{fList[20][[i,1,1]],fList[20][[i,1,2]],fList[20][[i,1,3]]},{fList[20][[i,2,2]]-
12,fList[20][[i,2,3]]-12,fList[20][[i,2,1]]+12}}];,{i,1,Length[fList[20]]}];
Do[
AppendTo[newList[21],{{fList[21][[i,1,1]],fList[21][[i,1,2]],fList[21][[i,1,3]]},{fList[21][[i,2,3]]-
12,fList[21][[i,2,1]],fList[21][[i,2,2]]}}];,{i,1,Length[fList[21]]}];
Do[
AppendTo[newList[22],{{fList[22][[i,1,1]],fList[22][[i,1,2]],fList[22][[i,1,3]]},{fList[22][[i,2,3]],fList[22][[i,2,1]]+12
Do[
AppendTo[newList[23],{{fList[23][[i,1,1]],fList[23][[i,1,2]],fList[23][[i,1,3]]},{fList[23][[i,2,3]]-
12,fList[23][[i,2,1]],fList[23][[i,2,2]]+12}}];,{i,1,Length[fList[23]]}];
Do[
```

```
AppendTo[newList[24],{{fList[24][[i,1,1]],fList[24][[i,1,2]],fList[24][[i,1,3]]},{fList[24][[i,2,3]]-
12,fList[24][[i,2,1]]+12,fList[24][[i,2,2]]+12}}];,{i,1,Length[fList[24]]}];
Do[
AppendTo[newList[25],{{fList[25][[i,1,1]],fList[25][[i,1,2]],fList[25][[i,1,3]]},{fList[25][[i,2,1]]-
12,fList[25][[i,2,2]],fList[25][[i,2,3]]+12}}];,{i,1,Length[fList[25]]}];
Do[
AppendTo[newList[26],{{fList[26][[i,1,1]],fList[26][[i,1,2]],fList[26][[i,1,3]]},{fList[26][[i,2,1]]-
12,fList[26][[i,2,2]]+12,fList[26][[i,2,3]]+12}}];,{i,1,Length[fList[26]]}];
Do[
AppendTo[newList[27],{{fList[27][[i,1,1]],fList[27][[i,1,2]],fList[27][[i,1,3]]},{fList[27][[i,2,1]]-
12,fList[27][[i,2,2]]-12,fList[27][[i,2,3]]+12}}];,{i,1,Length[fList[27]]}];
,{i,1,1}]
Do[pointList[i]={},{i,1,27}]
Do[Do[AppendTo[pointList[i],newList[i][[j,2]]],{j,1,Length[newList[i]]}],{i,1,27}]
allPoints={};
Do[AppendTo[allPoints,pointList[i]],{i,1,27}]
Show[Graphics3D[{Opacity[0.1],Cuboid[{0,0,0},{12,12,12}]}],Graphics3D[{Opacity[0.1],Cuboid[{0,0,0},{-
12,-12,-12}]}],Graphics3D[{Opacity[0.1],Cuboid[{0,0,0},{12,12,12}]}],Graphics3D[{Opacity[0.1],Cuboid[{24,24,2
12,-12,24}]}],(*Graphics3D[{Opacity[0.1],Cuboid[{0,12,0},{-12,24,-12}]}],*)Graphics3D[{Opacity[0.1],Cuboid[{1
12,-12}]}],Graphics3D[{Opacity[0.1],Cuboid[{12,12,0},{24,24,-12}]}],Graphics3D[{Opacity[0.1],Cuboid[{0,12,12}
12,24,24}]}],(*Graphics3D[{Opacity[0.1],Cuboid[{0,0,0},{12,12,12}]}],Graphics3D[{Opacity[0.1],Cuboid[{0,0,0},
(*combine points*)
firstPointList={}
Do[
Do[If[!MemberQ[Mod[pointList[j][[i]],12],0]Mod[pointList[j][[i]][[1]],12]≠ Mod[pointList[j][[i]][[2]], 12] ≠
Mod[pointList[j][[i]][[3]], 12], Print["F[", j, "]", "-- > ", pointList[j][[i]]]; AppendTo[firstPointList, pointList[
Do[v[i]={},{i,1,Length[firstPointList]}]
Do[AppendTo[v[i],firstPointList[[i]]-Mod[firstPointList[[i]],12]];
If[Mod[firstPointList[[i,1]],12]>Mod[firstPointList[[i,2]],12]Mod[firstPointList[[i,1]],12]>Mod[firstPointList[[i,3
AppendTo[v[i],firstPointList[[i]]+{12-Mod[firstPointList[[i,1]],12],-Mod[firstPointList[[i,2]],12],-
Mod[firstPointList[[i,3]],12]}];];
If[Mod[firstPointList[[i,2]],12]>Mod[firstPointList[[i,1]],12]Mod[firstPointList[[i,2]],12]>Mod[firstPointList[[i,3
AppendTo[v[i],firstPointList[[i]]+{-Mod[firstPointList[[i,1]],12],12-Mod[firstPointList[[i,2]],12],-
Mod[firstPointList[[i,3]],12]}];];
If[Mod[firstPointList[[i,3]],12]>Mod[firstPointList[[i,1]],12]Mod[firstPointList[[i,3]],12]>Mod[firstPointList[[i,2
AppendTo[v[i],firstPointList[[i]]+{-Mod[firstPointList[[i,1]],12],-Mod[firstPointList[[i,2]],12],12-
Mod[firstPointList[[i,3]],12]}];];
If[Mod[firstPointList[[i,1]],12]<Mod[firstPointList[[i,2]],12]Mod[firstPointList[[i,1]],12]<Mod[firstPointList[[i,3
AppendTo[v[i],firstPointList[[i]]+{-Mod[firstPointList[[i,1]],12],12-Mod[firstPointList[[i,2]],12],12-
Mod[firstPointList[[i,3]],12]}];];
If[Mod[firstPointList[[i,2]],12]<Mod[firstPointList[[i,1]],12]Mod[firstPointList[[i,2]],12]<Mod[firstPointList[[i,3
AppendTo[v[i],firstPointList[[i]]+{12-Mod[firstPointList[[i,1]],12],-Mod[firstPointList[[i,2]],12],12-
Mod[firstPointList[[i,3]],12]}];];
If[Mod[firstPointList[[i,3]],12]<Mod[firstPointList[[i,1]],12]Mod[firstPointList[[i,3]],12]<Mod[firstPointList[[i,2
AppendTo[v[i],firstPointList[[i]]+{12-Mod[firstPointList[[i,1]],12],12-Mod[firstPointList[[i,2]],12],-
Mod[firstPointList[[i,3]],12]}];];
AppendTo[v[i],firstPointList[[i]]+{12-Mod[firstPointList[[i,1]],12],12-Mod[firstPointList[[i,2]],12],12-
Mod[firstPointList[[i,3]],12]}],{i,1,Length[firstPointList]}]
```

```
Show[Graphics3D[Yellow,Opacity[0.1],GraphicsComplex[v[1],Polygon[{{1,2,3},{1,2,4},{1,3,4},{2,3,4}}]]}],(*G
6,7,8}]],*)Graphics3D[{Opacity[0.1],GraphicsComplex[v[6],Polygon[{{1,2,3},{1,2,4},{1,3,4},{2,3,4}}]]}],(*Graph
6,-5,8}]],*)Graphics3D[{Opacity[0.1],GraphicsComplex[v[7],Polygon[{{1,2,3},{1,2,4},{1,3,4},{2,3,4}}]]}],(*Grapl
6,-5,-4}]],*)Graphics3D[{Opacity[0.1],GraphicsComplex[v[8],Polygon[{{1,2,3},{1,2,4},{1,3,4},{2,3,4}}]]}],(*Grap
6,1,7}]],*)Graphics3D[{Opacity[0.1],GraphicsComplex[v[9],Polygon[{{1,2,3},{1,2,4},{1,3,4},{2,3,4}}]]}],(*Graph
6,2,13}]],*)Graphics3D[{Opacity[0.1],GraphicsComplex[v[11],Polygon[{{1,2,3},{1,2,4},{1,3,4},{2,3,4}}]]}],(*Gra
6,-4,7}]],*)Graphics3D[{Opacity[0.1],GraphicsComplex[v[13],Polygon[{{1,2,3},{1,2,4},{1,3,4},{2,3,4}}]]}],(*Gra
6,-5,1}]],*)Graphics3D[{Opacity[0.1],GraphicsComplex[v[15],Polygon[{{1,2,3},{1,2,4},{1,3,4},{2,3,4}}]]}],(*Gra
6,-5,13}]}],*)Graphics3D[{Opacity[0.1],GraphicsComplex[v[16],Polygon[{{1,2,3},{1,2,4},{1,3,4},{2,3,4}}]]}],(*G
6,1,2}]],*)Graphics3D[{Opacity[0.1],GraphicsComplex[v[17],Polygon[{{1,2,3},{1,2,4},{1,3,4},{2,3,4}}]]}],(*Grap
6,1,14}]],*)Graphics3D[{Opacity[0.1],GraphicsComplex[v[19],Polygon[{{1,2,3},{1,2,4},{1,3,4},{2,3,4}}]]}],(*Gra
6,13,14}]],*)
  Graphics3D[{Opacity[0.1],GraphicsComplex[{{-12,12,12},{0,12,12},{0,12,24},{0,24,24}},Polygon[{{1,2,3},{1,
6,13,14}]]*)(*,
  Graphics3D[{PointSize[.015],Point[pointList[10]]}],
  Graphics3D[{PointSize[.015],Point[pointList[11]]}],
  Graphics3D[{PointSize[.015],Point[pointList[15]]}],
  Graphics3D[{PointSize[.015],Point[pointList[16]]}],
  Graphics3D[{PointSize[.015],Point[pointList[19]]}],
  Graphics3D[{PointSize[.015],Point[pointList[25]]}],
  Graphics3D[{PointSize[.015],Point[pointList[26]]}],
  Graphics3D[{PointSize[.015],Point[pointList[27]]}]*)
  ]
```

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