Solving the Bioheat Equation for Transcutaneous Recharging of a Medical Device Using Electric Fields

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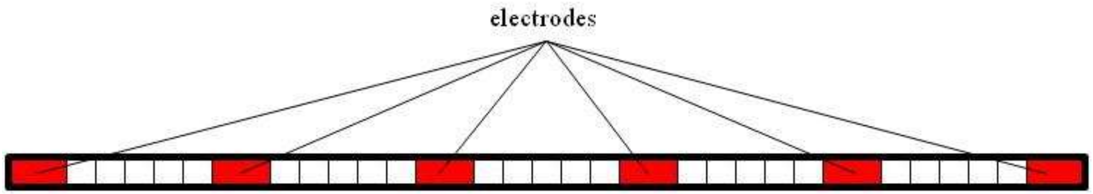
Electrophysiology

- The study of the body's electric activity
  - Can be small-scale (individual cells) or large-scale (entire organs)
- Electrophysiology often plays an important role in medical diagnostic procedures
  - Ex: ECG, EEG, EMG
- Signals often recorded by placing a series of electrodes on the surface of a patient's skin
  - Not always a practical approach—long-term collection of data may be required

A Possible Solution

- A subcutaneous (under the skin) recording device could remain in place semi-permanently
  - Device may be implanted almost anywhere in a minor surgical procedure



Schematic of proposed device

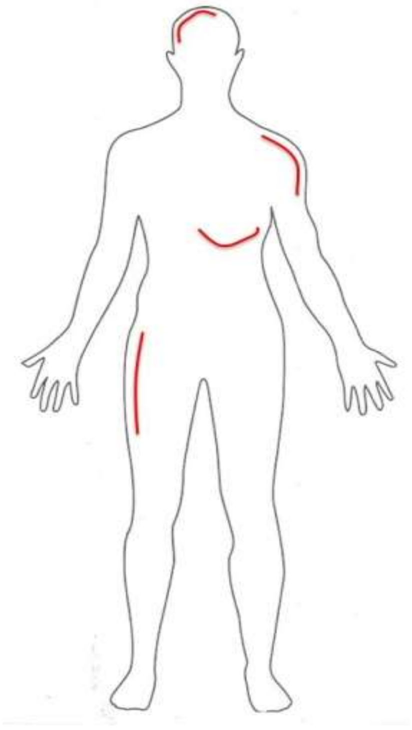
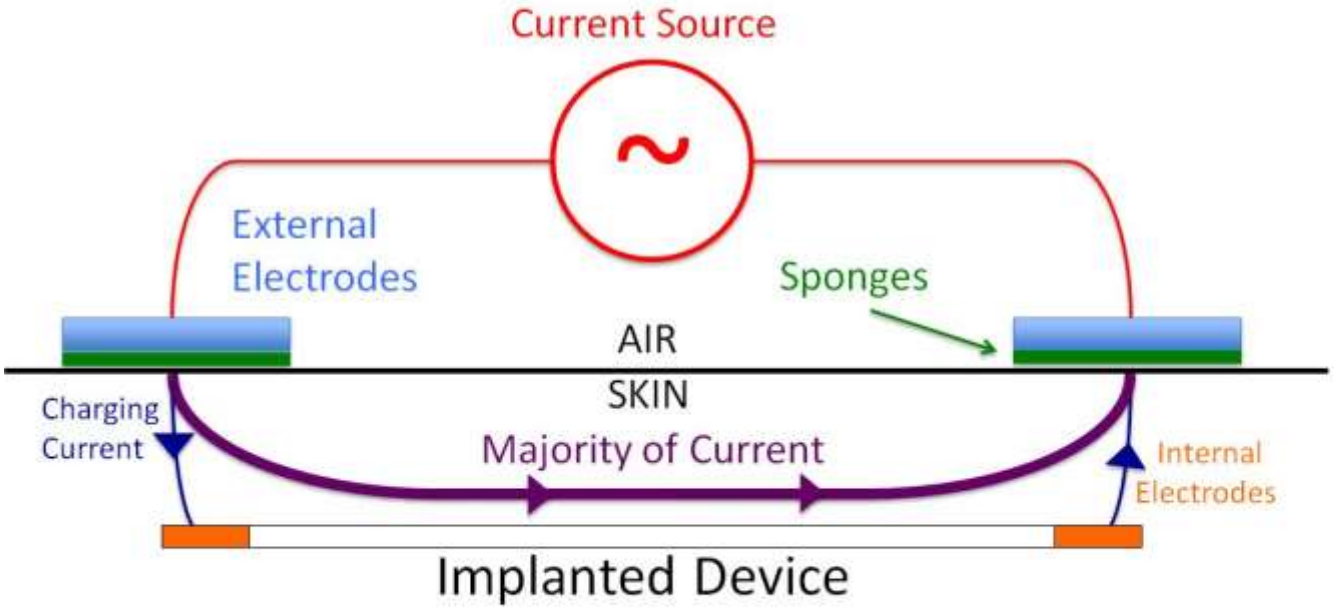


Figure created by Zachary Abzug

Transcutaneous Recharging

- Most implanted devices recharged via magnetic fields—not feasible for this device
- Instead, induce high frequency electric field using external source and sink electrodes



Transcutaneous recharging using electric fields

Figure created by Zachary Abzug

Upholding Medical Standards

- Problem with recharging via electric fields: current passing through tissue can cause thermal damage (Joule heating)
  - A temperature increase ≤ 2 ^{o}C is within medical standards
- Previous work: perform finite element analysis to investigate expected temperature increase

Project Objective

- Derive a closed-form solution for the anticipated temperature increase
  - Primary motivation: improved understanding of physical parameters on temperature increase

The "Extended" Bioheat Equation

$$\rho C \frac{\partial T}{\partial t} = k\nabla^2 T + \rho_b C_b \omega_b (T_b - T) + Q_{met} + \boldsymbol{J} \cdot \boldsymbol{E}$$

(terms labelled: [1] $\rho C \frac{\partial T}{\partial t} = k\nabla^2 T$; [2] $\rho_b C_b \omega_b (T_b - T) + Q_{met}$; [3] $\boldsymbol{J} \cdot \boldsymbol{E}$)

- [1] <u>The heat equation</u>: describes variation of temperature in a region as a function of time
- [2] <u>Pennes' extension to the heat equation</u>: accounts for heat transfer due to perfusion (blood flow) and metabolic heat production
- [3] <u>Source term</u>: accounts for heating due to power dissipation in tissue

Power Dissipation

- Calculate the work done by electromagnetic forces on a charge Q moving some infinitesimal distance d**l**:

$$dW = \boldsymbol{F} \cdot d\boldsymbol{l} = Q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v}dt$$

$$dW = Q\boldsymbol{E} \cdot \boldsymbol{v}dt$$

- Make the substitutions $Q = \int_V \rho d\tau$ and $\boldsymbol{J} = \rho\boldsymbol{v}$
- Rate at which work is done on all charges in a volume: $\frac{dW}{dt} = \int_V (\boldsymbol{J} \cdot \boldsymbol{E}) d\tau$
- Therefore, $\boldsymbol{J} \cdot \boldsymbol{E}$ is the work done on a charge per unit volume per unit time

Simplifying the Bioheat Equation

- Must make several simplifications

$$\rho C\frac{\partial T}{\partial t} = k\nabla^2 T + \rho_b C_b \omega_b (T_b - T) + Q_{met} + \boldsymbol{J}\cdot\boldsymbol{E}$$

(the terms $\rho C\frac{\partial T}{\partial t}$, ω_b and Q_{met} are each crossed out by an arrow pointing to 0)

  - Steady-state solution($\frac{\partial T}{\partial t} = 0$)
  - Ignore perfusion($\omega_b = 0$)
  - Ignore metabolic heat production($Q_{met} = 0$)
- Final equation to solve:

$$\nabla^2 T = -\frac{1}{k}(\boldsymbol{J}\cdot\boldsymbol{E})$$

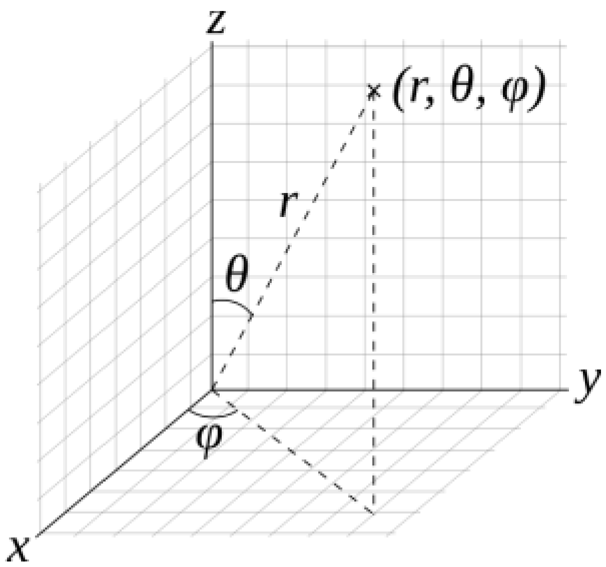
A Previous Solution

- Our method of solving the bioheat equation is similar to a solution given by Elwassif et al. [1]
  - Begin by relating the source term to the gradient of the electric potential

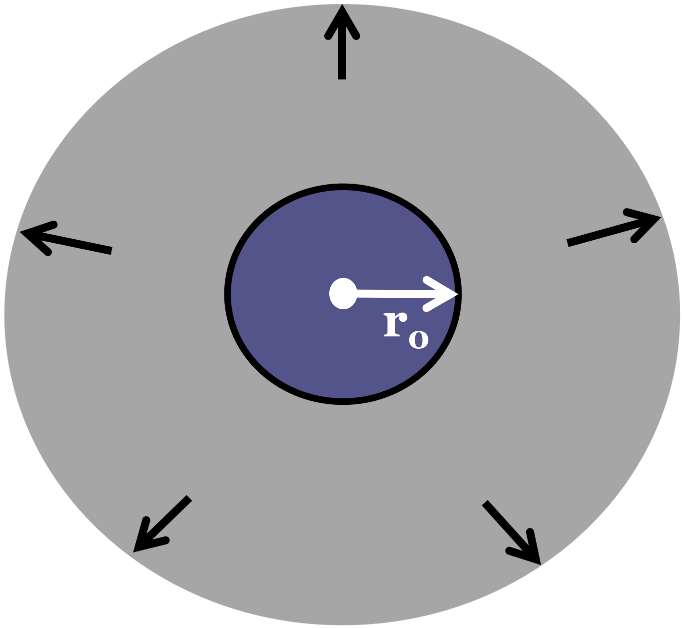
$$\boldsymbol{J} \cdot \boldsymbol{E} = \sigma \boldsymbol{E} \cdot \boldsymbol{E} = -\sigma \nabla V \cdot -\nabla V = \sigma |\nabla V||\nabla V| cos\theta = \sigma |\nabla V|^2$$

Geometric Considerations

- Treat electrode as a current-producing sphere in an infinite homogeneous and isotropic resistive material
- Second electrode is at infinity (V=0)



Spherical coordinate system
(image from Wikipedia)



Spherical electrode within infinite medium

A Solution in Spherical Coordinates

- Write bioheat equation in spherical coordinates
  - Ignore θ and φ dependence due to geometry

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = -\frac{\sigma}{k}|\nabla V|^2$$

- Must find an expression for the electric potential by solving Laplace's equation

Solving Laplace's Equation

- Laplace's equation is $\nabla^2 V = 0$
  - Write in spherical coordinates: $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0$
  - Multiply through by r²: $\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0$
  - Integrate twice with respect to r: $V(r) = -\frac{A}{r} + B$
  - Since zero potential at infinity, B=0: $V(r) = -\frac{A}{r}$

Solving Laplace's Equation

- To determine A, consider a point source of current in an infinite, homogeneous, isotropic medium.
  - The current density is: $\mathbf{J} = \frac{I}{4\pi r^2}\hat{\mathbf{r}}$
  - Since $\mathbf{J} = \sigma\mathbf{E} = -\sigma\nabla V$, the potential is:

$$\nabla V = -\frac{I}{4\pi\sigma r^2}\hat{\mathbf{r}} \Rightarrow \frac{dV}{dr} = -\frac{I}{4\pi\sigma r^2} \Rightarrow V(r) = \frac{I}{4\pi\sigma r}$$

  - Compare to $V(r) = -\frac{A}{r} \Rightarrow A = -\frac{I}{4\pi\sigma}$

Solving the Bioheat Equation

- Plug solution for potential into bioheat equation:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) = -\frac{\sigma}{k}\left|\nabla\left(-\frac{A}{r}\right)\right|^2$$

product rule for derivatives

take gradient with respect to r

$$\frac{2}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} = -\left(\frac{I}{4\pi}\right)^2\frac{1}{k\sigma r^4}$$

$$2r^3\frac{\partial T}{\partial r} + r^4\frac{\partial^2 T}{\partial r^2} = -\left(\frac{I}{4\pi}\right)^2\frac{1}{k\sigma}$$

- Make substitutions for clarity:

$$x^4 y'' + 2x^3 y' = c$$

Solving the Bioheat Equation

- The result is a second-order, linear, non-homogeneous differential equation

$$x^2 y'' + 2xy' = \frac{c}{x^2}$$

- The general solution to this form of DE is

$$y = y_c + y_p$$

complimentary function

Determine by solving the homogeneous case

particular solution

Determine using variation of parameters

The Complimentary Function

- The homogeneous case is: $x^2y'' + 2xy' = 0$
- This is a Cauchy-Euler equation, so the solution is of the form $y = x^m$
- Plug in y' and y'' to the homogeneous case and solve for m:

$$x^2m(m-1)x^{m-2} + 2xmx^{m-1} = 0$$

$$\Downarrow$$

$$m = 0 \quad m = -1$$

- Since $y = x^m$, the complimentary function is given by:

$$y_c = c_1y_1 + c_2y_2$$

$$y_c = \alpha x^0 + \beta x^{-1}$$

$$\boxed{y_c = \alpha + \frac{\beta}{x}}$$

The Particular Solution

- To use variation of parameters, rewrite as:

$$y'' + \frac{2}{x}y' = \frac{c}{x^4}$$

- The solution is given by $y_p = u_1 y_1 + u_2 y_2$, where y_1 and y_2 are from the complimentary function and

$$u_1 = \int u_1' = \int \frac{W_1}{W} \qquad u_2 = \int u_2' = \int \frac{W_2}{W} \qquad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} \qquad f(x) = \frac{c}{x^4}$$

$$y_p = \frac{c}{2x^2}$$

The General Solution

- The general solution is the sum of the complimentary function and the particular solution:

$$y = y_c + y_p = \alpha + \frac{\beta}{x} + \frac{c}{2x^2}$$

$$T(r) = \alpha + \frac{\beta}{r} + \frac{c}{2r^2}$$

$$\boxed{T(r) = \alpha + \frac{\beta}{r} - \left(\frac{I}{4\pi}\right)^2 \frac{1}{2k\sigma r^2}}$$

- Plugging this back into the bioheat equation verifies that it is a solution

The General Solution

- The solution is also valid in terms of units

Quantity	Unit
I	A
k	A/V*m
σ	V*A/m*K
α	K
β	K*m

- Need to determine α and β

Determining α

- Assume the tissue is unaffected by heating at an infinite distance from the electrode
  - As $r \to \infty$, $T(r) \to$ 310.15 K (body temperature)

$$T(r) = \alpha + \frac{\beta}{r} - \left(\frac{I}{4\pi}\right)^2 \frac{1}{2k\sigma r^2}$$

(the terms $\frac{\beta}{r}$ and $\left(\frac{I}{4\pi}\right)^2 \frac{1}{2k\sigma r^2}$ are struck through with arrows pointing to 0, 0) ⇒ α= 310.15 K

$$T(r) = 310.15 + \frac{\beta}{r} - \left(\frac{I}{4\pi}\right)^2 \frac{1}{2k\sigma r^2}$$

Determining β

- To determine a numeric value for β, we will need another boundary condition
  - Allow heat to exit system (otherwise temperature will rise indefinitely)
  - Assume heat cannot leave system at $r = r_o$
    - Use Fourier's Law: $\boldsymbol{q} = -k\nabla T$ (**q**=local heat flux)
    - If no heat can flow at $r = r_o$, then:

$$0 = \nabla T$$

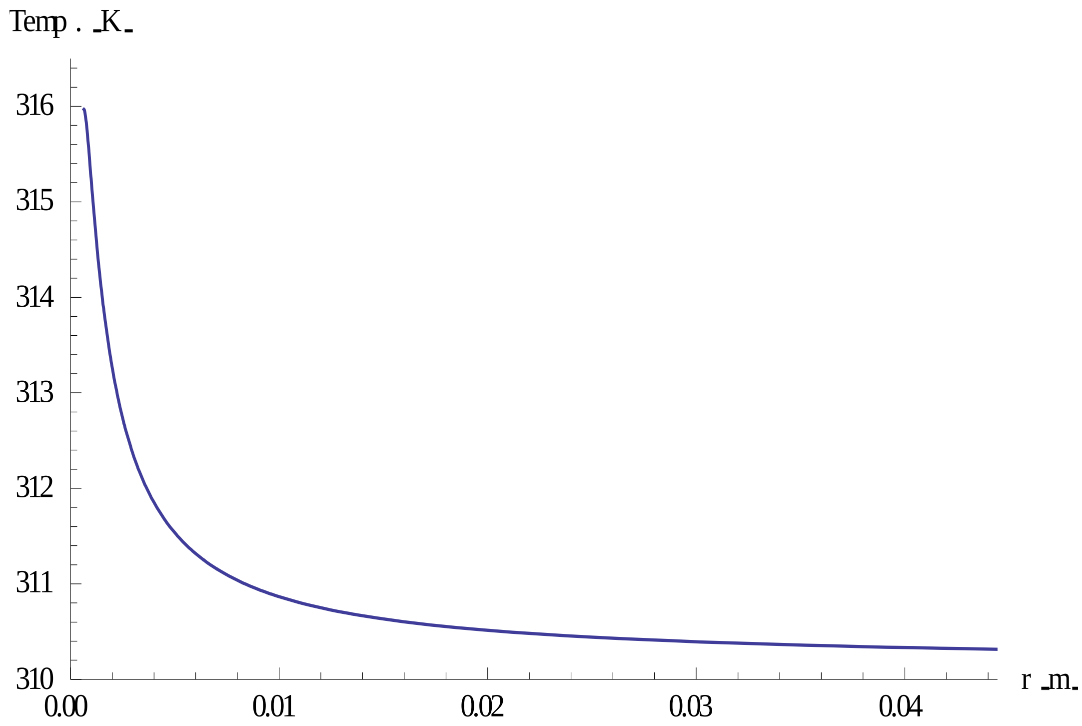
$$0 = -\frac{\beta}{r^2} - \frac{c}{r^3}$$

$$\beta = \left(\frac{I}{4\pi}\right)^2 \frac{1}{k\sigma r_0}$$

Determining β

- The solution for temperature is:

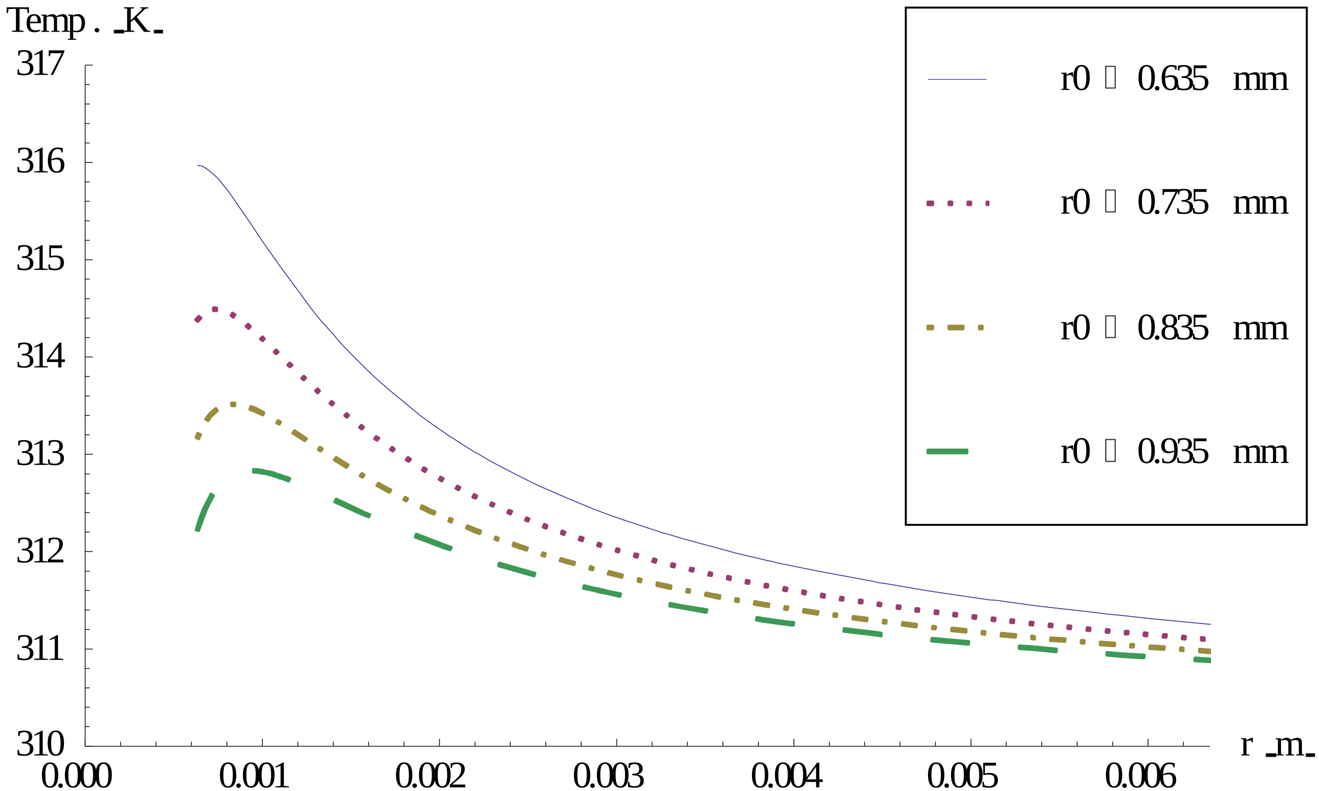
$$T(r) = 310.15 + \left(\frac{I}{4\pi}\right)^2 \frac{1}{k\sigma r_0}\frac{1}{r} - \left(\frac{I}{4\pi}\right)^2 \frac{1}{2k\sigma r^2}$$



A plot of temperature vs. radial distance from electrode (I = 11.7 mA, σ = 0.327 A/V*m, k = 0.565 W/m*K, r_o = 0.635 mm).

Sensitivity to r_0

- Behavior of solution is highly dependent on r_o



Dependence of temperature on r_o

Future Work

- What is the *physical* meaning of the solution?
- The temperature distribution is

$$T(r) = 310.15 + \left(\frac{I}{4\pi}\right)^2 \frac{1}{k\sigma r_0}\frac{1}{r} - \left(\frac{I}{4\pi}\right)^2 \frac{1}{2k\sigma r^2}$$

or

$$T(r) = 310.15 + \frac{A}{r_0 r} - \frac{A}{2r^2} \quad \text{(where } A = \left(\frac{I}{4\pi}\right)^2 \frac{1}{k\sigma}\text{)}$$

- What does it mean to have two similar terms competing?

Conclusions

- Recharging a subcutaneous medical device using electric fields can increase tissue temperature
- We show that the steady-state temperature distribution is given by $T(r) = 310.15 + \frac{\beta}{r} - \left(\frac{I}{4\pi}\right)^2 \frac{1}{2k\sigma r^2}$
- Future work: investigate how physical parameters influence temperature increase

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- Duke University
  - Tom Jochum
  - Zachary Abzug
  - Dr. Patrick Wolf

References

[1] Elwassif, Maged M., Qingjun Kong, Maribel Vazquez, and Marom Bikson, “Bio-heat transfer model of deep brain stimulation-induced temperature changes,” *J. Neural Eng*. **3**, 306-315, Nov. 2006

[2] Griffiths, David J. "Conservation Laws." *Introduction to Electrodynamics*. Upper Saddle River, NJ: Prentice Hall, 1999. 345-46. Print.

[3] Malmivuo, Jaakko and Robert Plonsey. *Bioelectromagnetism: Principles and Applications of Bioelectric and Biomagnetic Fields*. New York: Oxford University Press, 1995. Bioelectromagnetism. N.p., n.d. Web. 2 April 2012. http://www.bem.fi/book/index.htm.

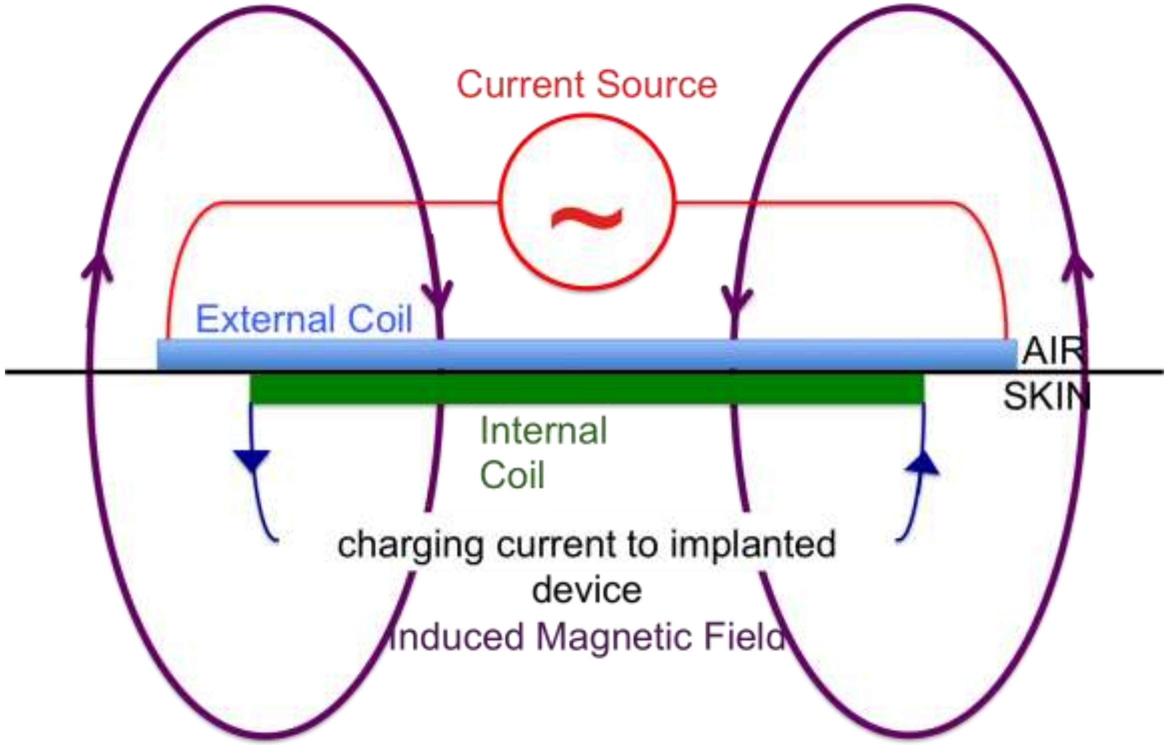
[4] Wissler, Eugene H., “Pennes’ 1948 paper revisited,” *J. Appl. Physiol.* ***85***, 35-41, 1998

Questions?

Extra Slides

Transcutaneous Recharging

- Most implanted devices recharged via magnetic fields—not feasible for this device



Transcutaneous recharging using magnetic fields

Figure created by Zachary Abzug

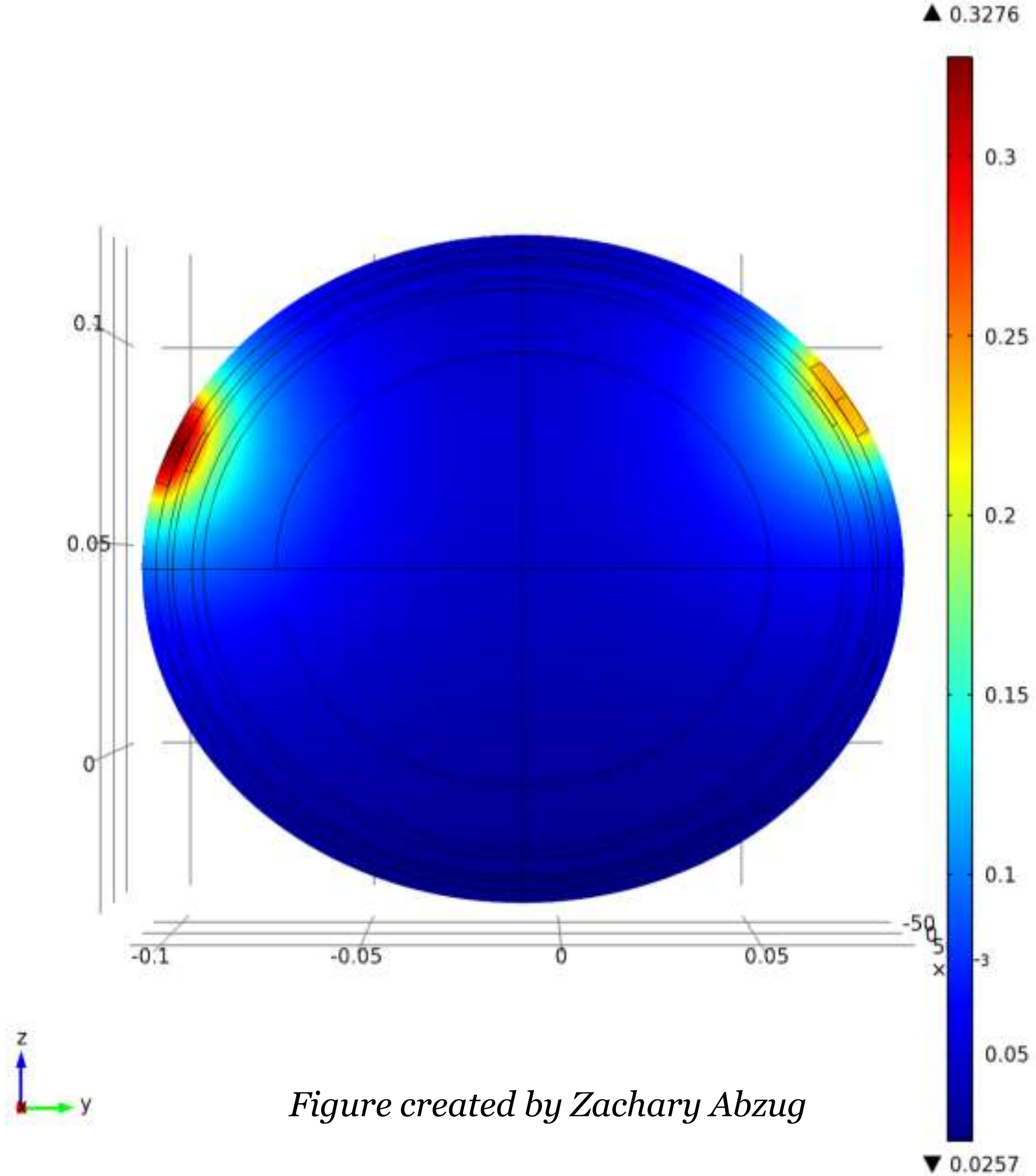


Figure created by Zachary Abzug

The Heat Equation

- The heat equation describes the three-dimensional variation of temperature in a region as a function of time

$$\rho C \frac{\partial T}{\partial t} = k \nabla^2 T$$

ρ = density
C = specific heat
k = thermal conductivity

- Not a complete model of heat transfer in biological situations due to perfusion (blood flow)

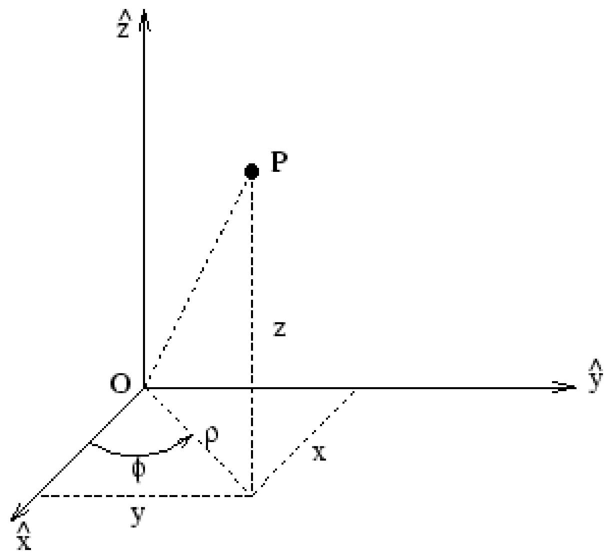
The Bioheat Equation

- The rate of heat transfer between blood and tissue is proportional to:
  - The volumetric perfusion rate
  - The difference between the arterial blood temperature and the local temperature
- Also add term (Q_{met}) to account for metabolic heat production
- The bioheat equation is:

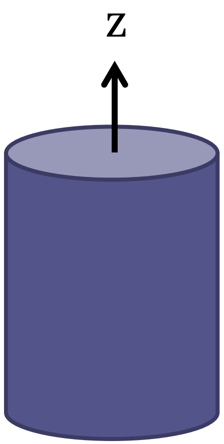
$$\rho C \frac{\partial T}{\partial t} = k \nabla^2 T + \rho_b C_b \omega_b (T_b - T) + Q_{met}$$

ρ_b = density of blood
C_b = specific heat of blood
T_b = temperature of blood
ω_b = perfusion rate per unit volume of tissue
T = local tissue temperature

A Solution in Cylindrical Coordinates



Cylindrical coordinate system
(image from uic.edu)

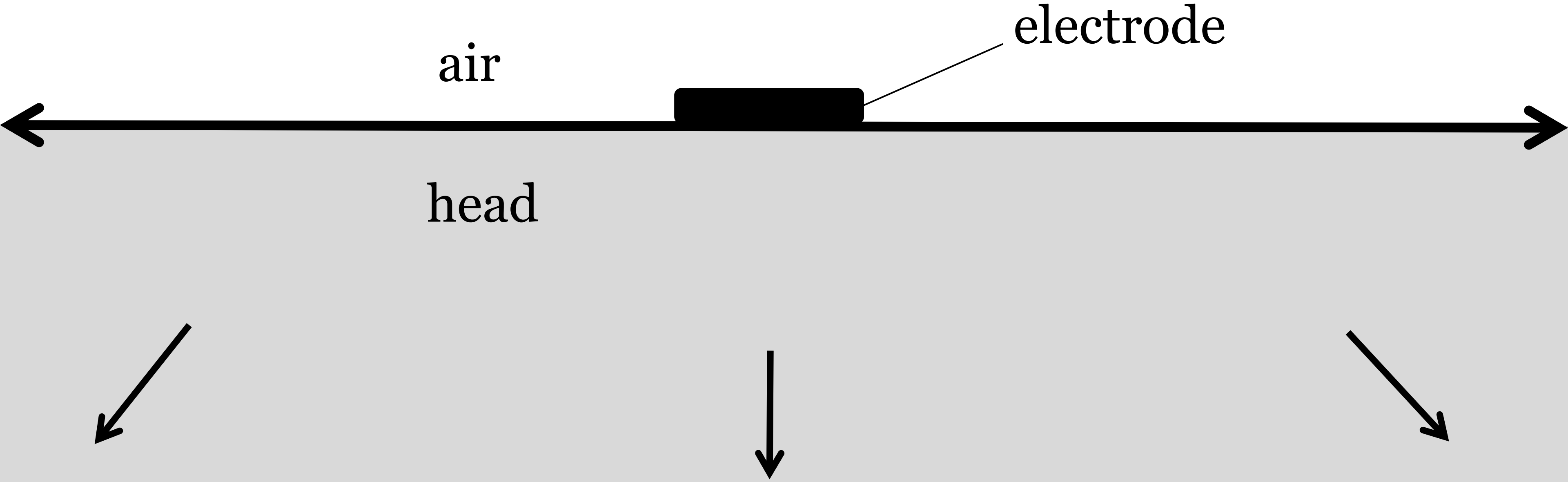


- Wrote Laplacian in cylindrical coordinates, ignoring φ and z dependence due to geometry

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = -\frac{\sigma}{k}|\nabla V|^2$$

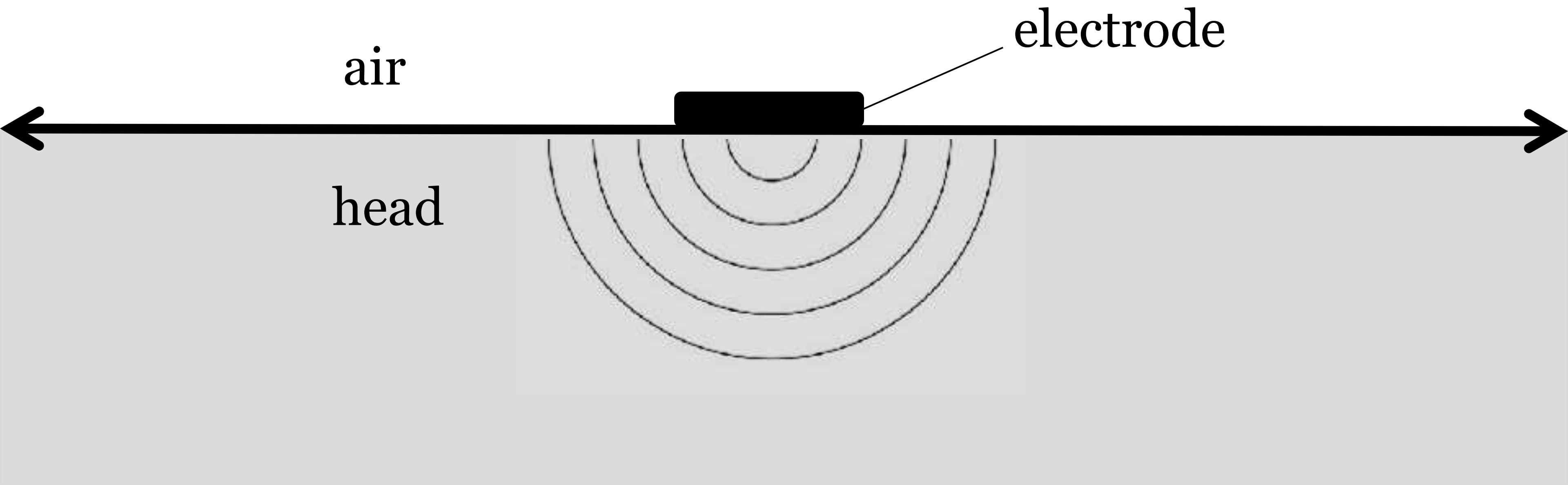
A Solution in Cylindrical Coordinates

- For our analysis, model head as infinitely wide and deep homogeneous resistive material
- Place one electrode on surface ($V=V_{applied}$) and one electrode at infinity ($V=0$)



A Solution in Cylindrical Coordinates

- It's acceptable to ignore φ dependence in our situation because of the axial symmetry
- Problem: we cannot ignore *z* dependence

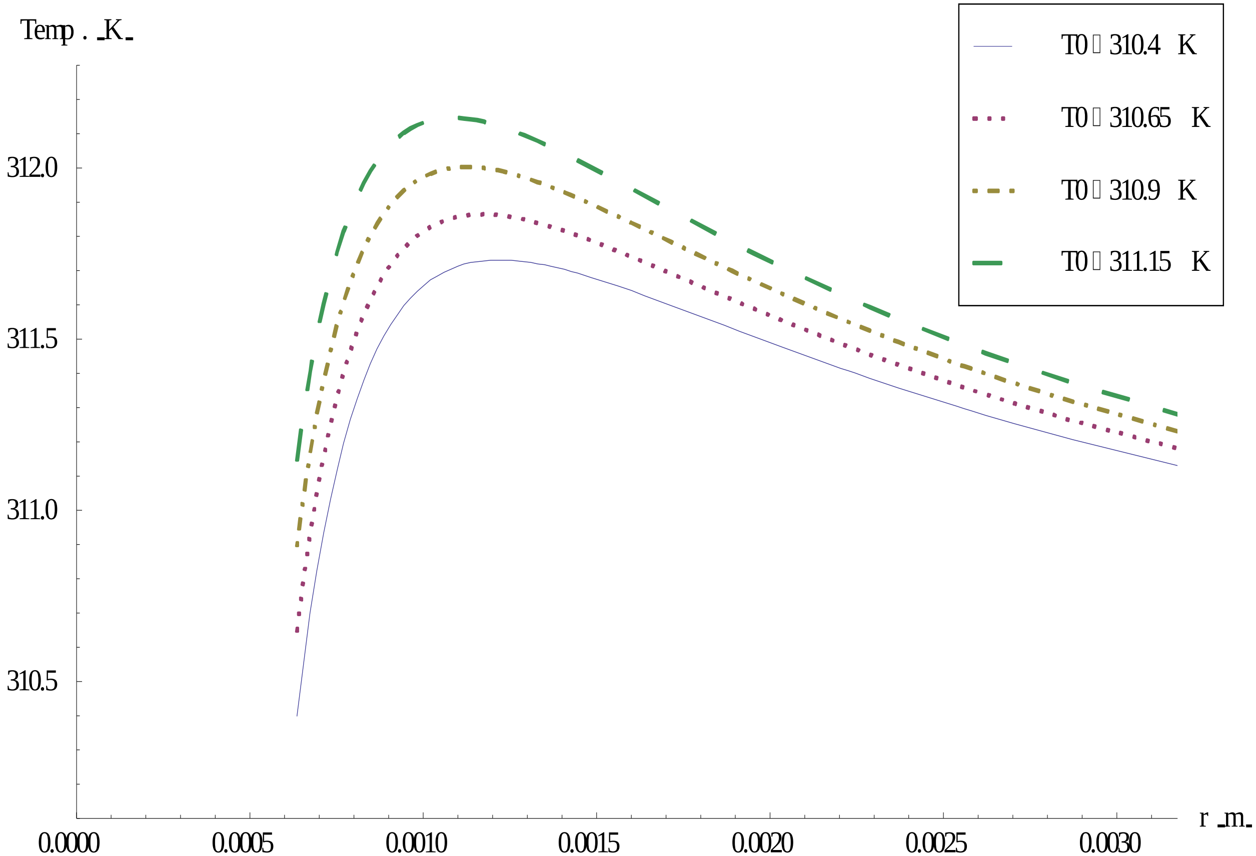


Determining β – Approach #1

- Want to know how the temperature behaves at the electrode's surface ($r=r_o$)
  - Requires knowing I, σ, k, r_o, and T_o (the temperature at the electrode's surface)
  - Choose representative values for I, σ, k, and r_o
  - Parameterize β based on selected values T_o
- Solve for β at r_o: $\left(T_0 - 310.15 + \left(\frac{I}{4\pi}\right)^2 \frac{1}{2k\sigma r_0{}^2}\right) r_0 = \beta$
- The solution for temperature:

$$T(r) = 310.15 + \left(T_0 - 310.15 + \left(\frac{I}{4\pi}\right)^2 \frac{1}{2k\sigma r_0{}^2}\right)\frac{r_0}{r} - \left(\frac{I}{4\pi}\right)^2 \frac{1}{2k\sigma r^2}$$

Determining β – Approach #1



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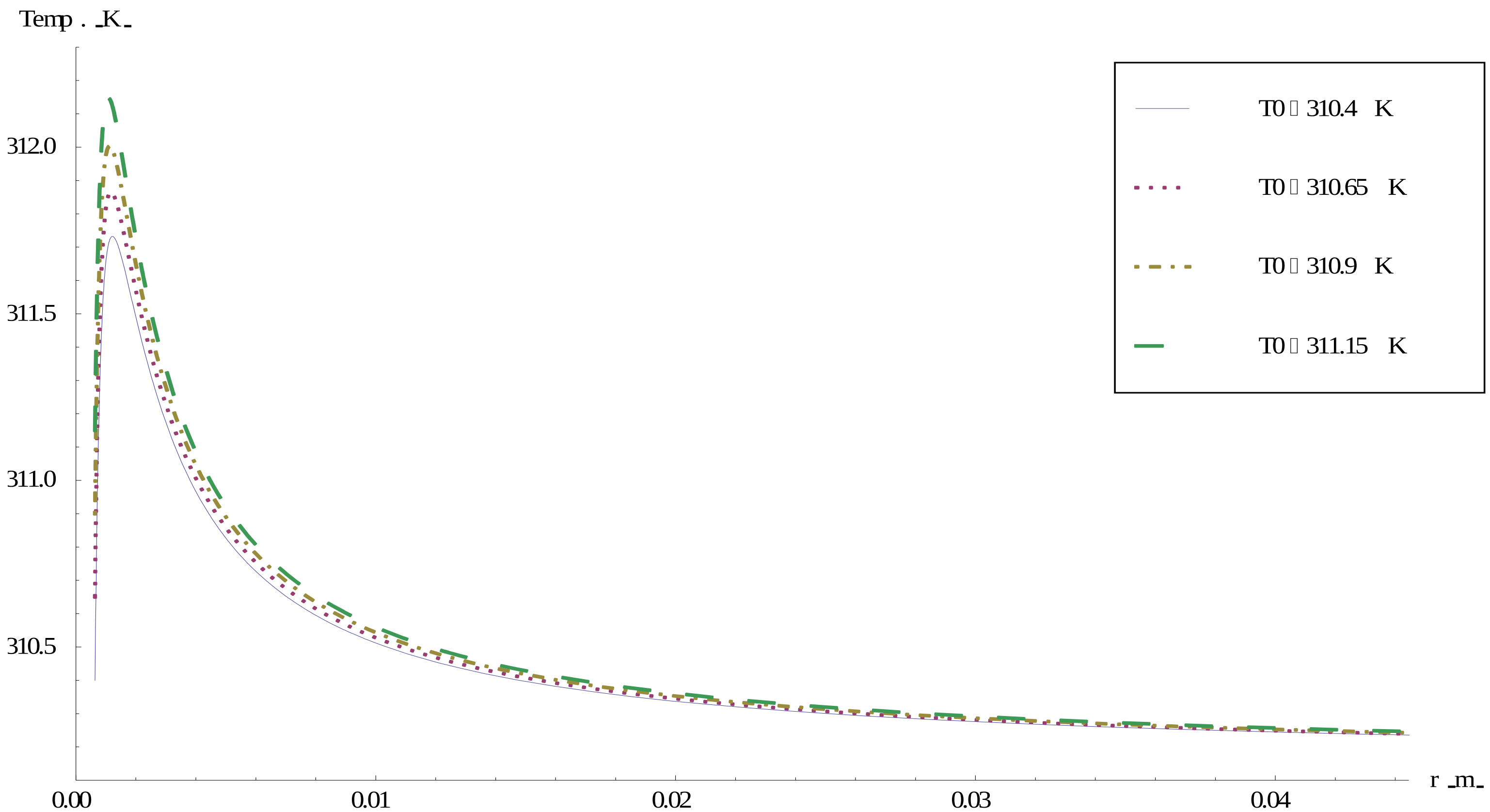
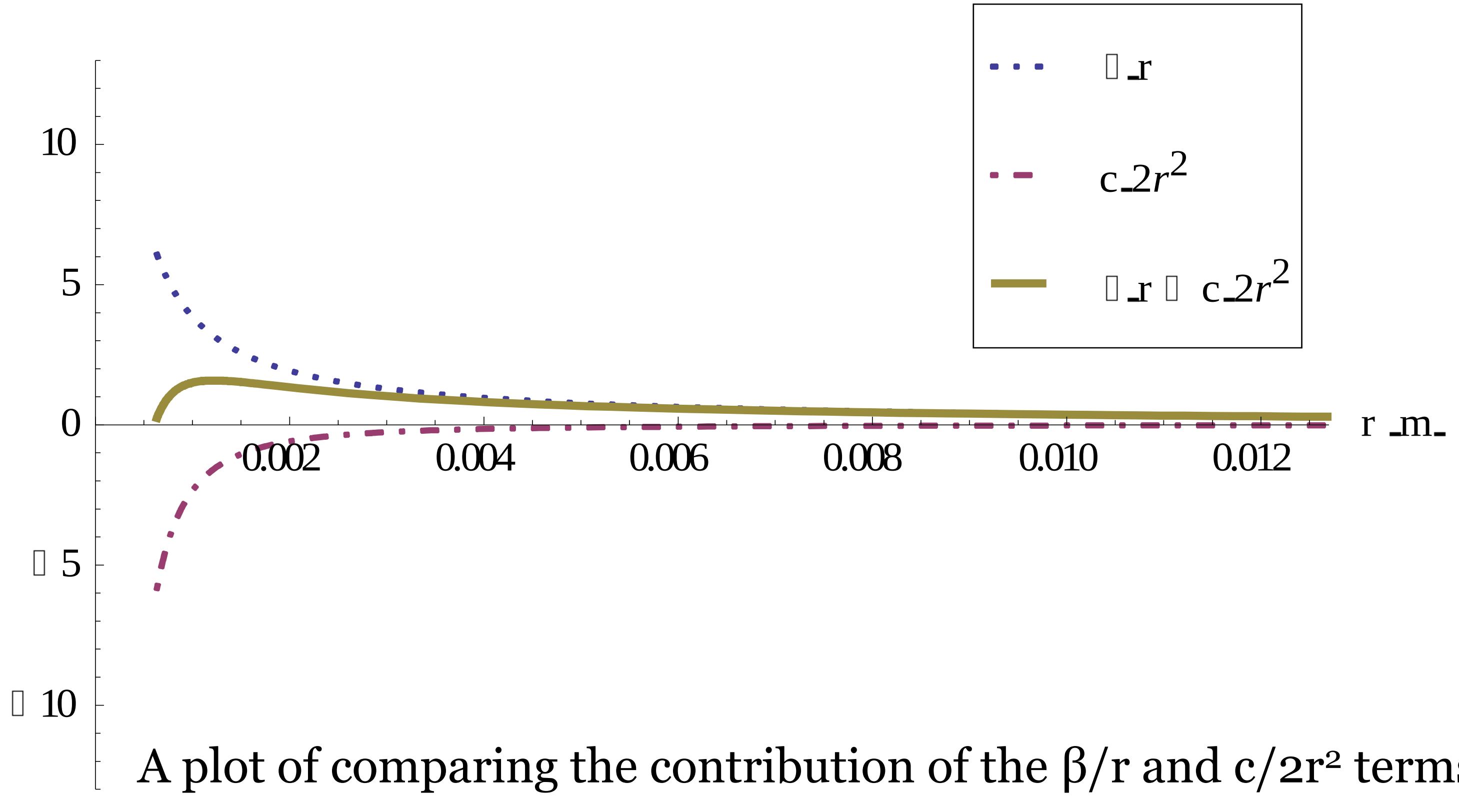
Determining β – Approach #1



A plot of temperature vs. radial distance from electrode (I = 11.7 mA, σ = 0.327 A/V*m, k = 0.565 W/m*K, r_o = 0.635 mm).

Temperature Peak

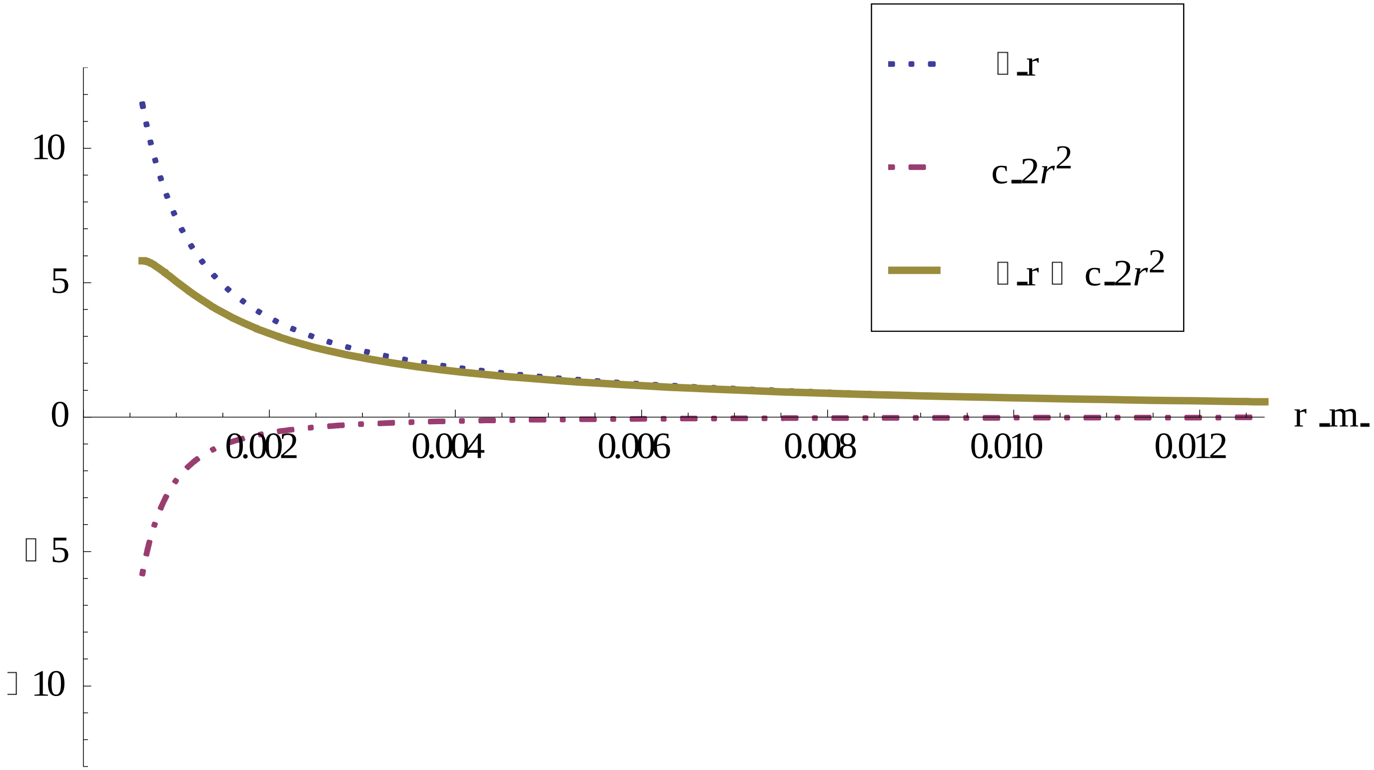
- If β is a smaller value, we see an initial peak in temperature
- Remember solution is of the form: $T(r) = \alpha + \frac{\beta}{r} + \frac{c}{2r^2}$



A plot of comparing the contribution of the β/r and c/2r² terms

Temperature Peak

- Would like to account for the peak in temperature
- Remember solution is of the form: $T(r) = \alpha + \frac{\beta}{r} + \frac{c}{2r^2}$



A plot of comparing the contribution of the β/r and $c/2r^2$ terms