

The Evaluated Burau Representation of the Braid Group

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Abstract

It was known that the Burau representation is faithful on the braid group of n strands where $n \leq 3$ [13]. Then, Stephan Bigelow showed this map is unfaithful for a braid Group of n strands when $n \geq 5$ [9]. Currently, whether the Burau representation is faithful when $n = 4$ is still an open question. In this paper, we study the kernel of the Burau representation evaluated at the τ^{th} root of unity. The main result of this paper is when the Burau representation of the braid group of n strands is evaluated at any $\tau^{th} \neq 3$ root of unity the representation is unfaithful.

1 Introduction

The disk of $2g+1$ marked points is the quotient by the hyperelliptic involution, of closed orientable surface of genus g . The surface serves as a branched covering space of the disk of n marked points. These marked points are better defined as branch points (*the points fixed by the group action*). Not only are the surfaces related by a branched cover, but under the Birman-Hilden theorem the mapping class group of the disk is isomorphic to a subgroup of the mapping class group of χ [Figure 1]. The mapping class group of the disk of n marked points is also isomorphic to the braid group of n strands. The braid group can be studied using representation theory, as a group the representation is a subset of invertible matrices. Specifically, we will use the Burau Representation which maps the braid group of n strands to $GL_n(\mathbb{Z}[t, t^{-1}])$. Evaluating t yields the evaluated Burau representation. In this paper, we study the kernel of the Burau representation evaluated at the τ^{th} root of unity. We will show that the evaluated Burau representation is an unfaithful map because its kernel is non-trivial.

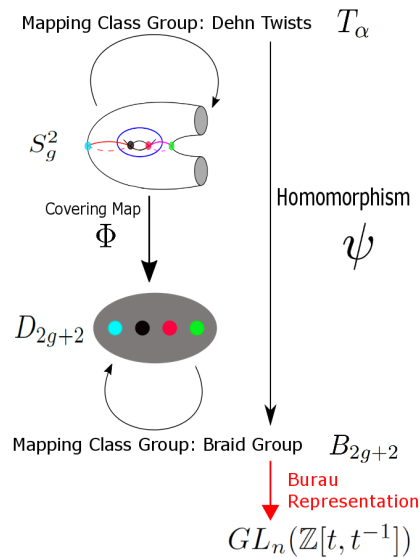


Figure 1: A representation of maps between topological surfaces.

2 Background

2.1 Surface of Genus g

A surface can be thought of more generally as a topological space.

Definition 1. Let X be a set. Let Ω be a collection of X 's subset, such that

1. The union of any collection of sets that are elements of Ω are a subset of Ω .

2. The intersection of any finite collection of sets that are elements of Ω are a subset of Ω .
3. The empty set, \emptyset and the whole X are a subset of Ω .

Together (X, Ω) form a **topological space** where the elements of X are the points of the topological space and the elements of Ω are the open sets of the topological space [8].

Definition 2. Let χ and χ' denote distinct surfaces. A **homeomorphism** is a bijective function $f : \chi \rightarrow \chi'$ where f and f^{-1} are continuous.

We can use homeomorphisms to define an equivalence relation on the set of orientable surfaces, where two surfaces are related if and only if they are homeomorphic. If we limit ourselves to orientable surfaces then homeomorphisms define the same equivalence classes as the relation determined by genus.

Definition 3. Let S^1 denote a one-sphere. A **torus** is the Cartesian product of two one-spheres (circles). This can be defined as $T = S^1 \times S^1$. [Figure 2(a)]

Definition 4. A **surface of genus g** is the connected sum of g tori. A surface of genus g is denoted χ_g . [Figure 2(b)]

We can also classify surfaces by the number of boundary components they contain.

Definition 5. A **surface with a boundary** is a a surface minus an open disk. A surface with genus g and b boundary components is denoted χ_g^b . [Figure 2(c)]

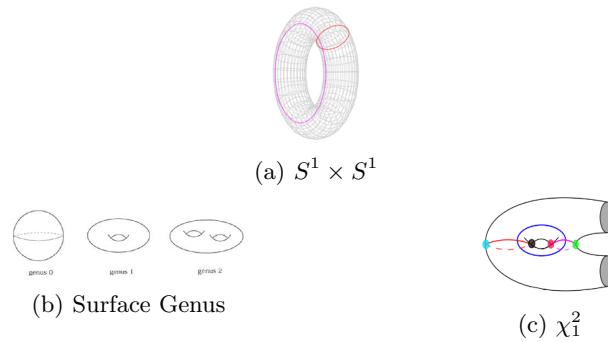


Figure 2: A few examples of surfaces.

2.2 Surfaces as a cover of disks

A cover of the topological space, χ' , consists of a topological space χ and a map Φ which maps open neighborhood of χ homeomorphically to χ' . The covering space, denoted by χ , is any topological space that can be mapped by a covering map to the topological space χ' .

A surface of genus g with one or two boundaries, χ_g^b (where $b \in \{1, 2\}$), and the hyperelliptic involution, ι , form the Birman-Hilden double (*i.e.* two fold) branched cover of D_{2g+b} [7]. The hyperelliptic involution is a map such that

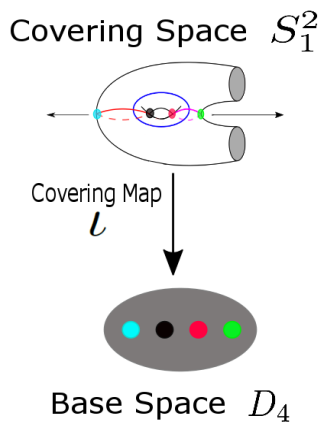


Figure 3: A cover of D_4 .

$\iota : \chi_g^b \rightarrow \chi_g^b$, defined by rotating the surface χ_g^b by π radians about the axis of rotation indicated in Figure 3 [4].

Notice, in Figure 3, that the axis of rotation intersects χ_1^2 at 4 points and the the base space is D_4 . These points are the branch points fixed by the group action. The quotient of χ_g^b by the hyperelliptic involution $\iota : \chi_g^b \rightarrow \chi_g^b$ is D_{2g+1} . Thus, χ_g^1 is the Birman-Hilden double covering space of the base space D_{2g+1} and χ_g^2 is the double branched covering space of the base space, D_{2g+2} .

2.3 Mapping Class Groups

The topological surfaces discussed in subsections 1 and 3 can be deformed and warped. The maps which deform the surface are called homeomorphisms. These maps are the focus of our study

Definition 6. An isotopy is a continuous function $I : \chi_g^b \times [0, 1] \rightarrow \chi_g^b$ where $I(x, 0) = f(x)$ and $I(x, 1) = g(x)$ such that for all $t \in [0, 1]$ the map $I(x, t)$ is a homeomorphism.

Two functions are isotopic if there exists an isotopy which maps one function to the other. Isotopy defines an equivalence relation on the set of homeomorphisms of χ_g^b (e.g. The identity homeomorphism is related to all rigid transformations under isotopy). If $h \in \text{Homeo}(\chi)$, we let $[h]$ denote the set of all homeomorphisms that are isotopic to h and we say that $[h]$ is the mapping class of the homeomorphism h . The set of all mapping classes of a surface χ_g^b under function composition is the mapping class group of χ_g^b denoted $\text{MOD}(\chi_g^b)$. An alternate description of the mapping class group is as a quotient group of the surface χ_g^b 's homeomorphisms. The set A modulo the normal subgroup of homeomorphisms isotopic to the identity is a factor group under function composition. This factor group is the mapping class group of a surface, χ_g^b . The results of the paper will be applicable to a subgroup of the $\text{MOD}(\chi_g^b)$ called the symmetric mapping class group, $\text{SMOD}(\chi_g^b)$. The subgroup's elements are the mapping classes which commute with the covering map, ι . (Note: $\text{MOD}(\chi_g^1) = \text{SMOD}(\chi_g^1)$)

2.4 Dehn Twists

The mapping class group of χ_g^b is generated by the Dehn twists under function composition. A Dehn twist is shown in Figure 4.

Definition 7. Using polar coordinates (r, θ) for points in the plane \mathbb{R}^2 , let A be an annulus made up of those points with $1 \leq r \leq 2$ whose core is the set of points where $r = \frac{3}{2}$. A **Dehn twist** is a map $T_A : (r, \theta) \rightarrow (r, \theta - 2r)$

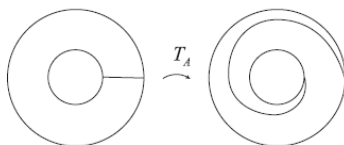


Figure 4: A Dehn twist on an annulus. [4]

Definition 8. A **simple closed curve** on a surface, χ , is the image of a circle, S^1 , in the surface, χ , under a continuous and injective function. [Figure 5(a)]

The neighborhood about any simple closed curve, α in χ is an annulus, A [Figure 5(b)]. Applying a Dehn twist to the annulus, A , with core α , is called a Dehn twist about α . Let T_α denote a Dehn twist about a curve, α .

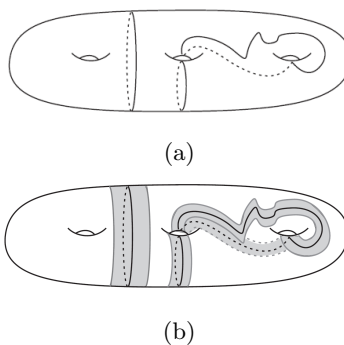


Figure 5: A set of simple closed curves (a) and their corresponding annuli (b)[4].

The action of the Dehn twist is well defined and injective, $T_\alpha = T_\gamma$ if and only if $[\alpha] = [\gamma]$. A Dehn twist along a simple closed curve α can be imagined as cutting, twisting and gluing. Recognize that any curve intersecting α acquires an extra twist about α [Figure 6].

Let α and γ denote curves from arbitrary isotopy classes of simple closed curves. we define the geometric intersection number, $i(\alpha, \gamma)$, as the total number of intersection points of α and γ . We'll distinguish two relations on the group generated by the Dehn twists.

1. The braid relation shows that for two curves, α and γ , $i(\alpha, \gamma) = 1$ if and only if $T_\alpha T_\gamma T_\alpha = T_\gamma T_\alpha T_\gamma$. The braid relation shows that deforming a surface with the composition $T_\alpha T_\gamma T_\alpha$ yields a surface that is homeomorphic to the surface deformed by the homeomorphism $T_\gamma T_\alpha T_\gamma$.

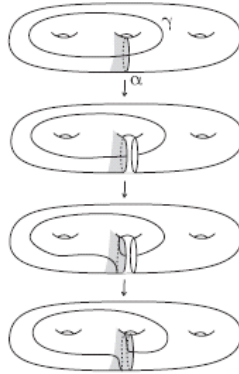


Figure 6: The first sequence is the cutting, the second sequence is the twisting and the third sequence is the gluing. Together they form a Dehn twist about curve α . [4]

2. The disjointness relation states that for two curves, α and γ , $i(\alpha, \gamma) = 0$ if and only if $T_\alpha T_\gamma = T_\gamma T_\alpha$. The Disjointness relation shows that if two curves, α and γ do not intersect then T_α and T_γ commute.

2.5 Homomorphism between Mapping Class Groups

Let B_n represent the braid group of n strands. Let $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ define a simple closed curve such that $\tilde{\alpha} \subset \chi_g^1$. Let α , β and γ be arcs in D_3 . We define a homomorphism $\psi : \text{MOD}(\chi_g^1) \rightarrow B_{2g+1}$, such that $\psi(T_{\tilde{\alpha}}) = \sigma_\alpha$ [Figure 7].

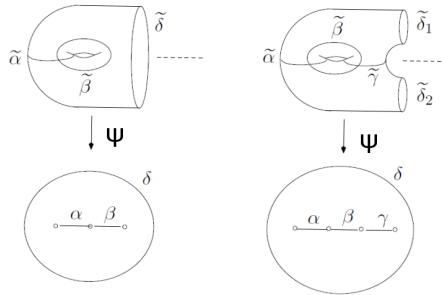


Figure 7: The homomorphism ψ maps elements contained in the covering space are denoted with a tilde and each corresponding element in the base space is denoted without a tilde.

The Birman-Hilden theorem states that for a surface, χ_g^1 , and a disk of $2g+1$ marked points, D_{2g+1} , $\text{SMOD}(\chi_g^1) \cong \text{MOD}(D_{2g+1})$. Furthermore, the theorem shows that for surfaces with two symmetric boundary components which are interchanged by the hyper elliptic involution, ι , then $\text{SMOD}(\chi_g^2) \cong B_{2g+2}$.

2.6 The Braid Group

The mapping class group of D_n is isomorphic to the braid group of n strands. For example, in Figure 6, a homeomorphism of D_3 is expressed as the combination of four generators. If left-right motion is conceptualized as backward and forward motion in time, respectively, then the strands are tracing the motion of the marked points in the disk through time. Furthermore, each cross in the strands is analogous to a generator of the braid group. Let B_n denote the braid group of n strands. Let $\sigma_1, \dots, \sigma_{n-1}$ denote the standard generators of B_n and let $\sigma_1^{-1}, \dots, \sigma_{n-1}^{-1}$ denote their inverses [Figure 8(a)]. The Artin representation of the braid group of n -strands is defined $B_n = \{\sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| > 1, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } 1 \leq i \leq n - 2\}$.

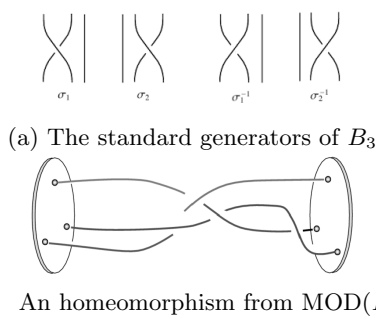


Figure 8: Expressing elements of $\text{SMOD}(D_3)$ as elements of B_3 . [10][11]

In Artin's presentation of the braid group there are 2 relations defined on the group. They are:

1. The braid relation states that $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ for $1 \leq i \leq n - 2$.
2. The Disjointness relation states that $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| > 1$. This means that for two braids that do not share a strand the braids are commutative.

Notice that Artin's braid relations are similar to the relations defined on the Dehn twist. The map ψ transfers the braid relation and the Disjointness relation from $\text{MOD}(\chi_g^1)$ to the braid group of $2g + 1$ marked points.

3 Burau Representation

3.1 Introduction

Let I_k denote the $k \times k$ identity matrix. Then the Burau representation of the braid group is the map:

$$\rho : B_n \rightarrow GL_n(\mathbb{Z}[t, t^{-1}]) \text{ defined by } \sigma_i \rightarrow I_{i-1} \oplus \begin{bmatrix} 1-t & t \\ 1 & 0 \end{bmatrix} \oplus I_{n-i-1}.$$

The Burau representation has a probabilistic interpretation. Vaughan Jones describes the Burau representation of the braid group as a matrix representation of stochastic processes. Given a positive braid σ on n strands, interpret it as a set of n intertwining hiking trails which cross at bridges. Then at every

intersection a traveler crossing the bridge has a probability t of falling through the bridge and continuing along the lower trail. Thus, the $(i, j)^{th}$ entry of the Burau representation of any given σ is the probability that a traveler who starts walking on the i^{th} trail will end in the j^{th} trail [12].

Birman showed the Burau representation is faithful on B_n when $n \leq 3$ [13]. Then in 1999, Stephan Bigelow showed this map is unfaithful for B_n when $n \geq 5$ [9]. However, whether the Burau representation is faithful when $n = 4$ is still an open question. If t is evaluated at some value (e.g. *Integral Burau representation is at $t = -1$*) then we call the Burau representation the “evaluated Burau representation”.

3.2 Algebraic Structures under the Burau Representation

We study the kernel of the Burau representation using the algebraic structures created by the image of $\sigma_1\sigma_2\sigma_1$, ζ , under the evaluated Burau representation. The element is notable because $\sigma_1\sigma_2\sigma_1$ is specified in Artin’s presentation of the braid group to define the braid relation. Other than specifying an element, the relation itself is not used to produce our results. Instead we will observe the general form that arises when we raise this matrix to the k^{th} power. The element ζ can be expressed as:

$$\rho(\sigma_1\sigma_2\sigma_1) = \zeta = \begin{bmatrix} 1-t & (1-t)t & t^2 \\ 1-t & t & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Definition 9. Let ϕ_i be the i^{th} cyclotomic polynomial and $Z = \{d \in \mathbb{N} \mid d \text{ divides } m+2\}$. Define the polynomial \mathbf{a}_m such that:

$$a_m = \begin{cases} \frac{1+t^{m+1}+t^{m+2}}{\phi_3}, \forall m \equiv 0 \pmod{3} \\ -\frac{1}{\phi_3} \prod_{d \in Z} \phi_d = \frac{1-t^{m+2}}{\phi_3}, \forall m \equiv 1 \pmod{3} \\ \frac{1+t^m+t^{m+2}}{\phi_3}, \forall m \equiv 2 \pmod{3} \end{cases}$$

Lemma 1. If $k \in \mathbb{N}$ such that k is even then the image of $(\sigma_1\sigma_2\sigma_1)^k$ in B_3 ,

under the Burau representation is $\zeta^k = \begin{bmatrix} a_{\frac{3}{2}k} & ta_{\frac{3}{2}k-2} & t^2a_{\frac{3}{2}k-2} \\ a_{\frac{3}{2}k-2} & ta_{\frac{3}{2}k-1} & t^2a_{\frac{3}{2}k-2} \\ a_{\frac{3}{2}k-2} & ta_{\frac{3}{2}k-2} & t^2a_{\frac{3}{2}k-3} \end{bmatrix}.$

Proof. Assume $k \in \mathbb{N}$ such that k is even. We will prove the form above with induction. Consider the base case where $k = 2$, then:

$$\zeta^2 = \begin{bmatrix} \frac{1}{\phi_3}(1+t^4+t^5) & \frac{1}{\phi_3}(t(1-t^3)) & \frac{1}{\phi_3}(t^2(1-t^3)) \\ \frac{1}{\phi_3}(1-t^3) & \frac{1}{\phi_3}(t+t^3+t^5) & \frac{1}{\phi_3}(t^2(1-t^3)) \\ \frac{1}{\phi_3}(1-t^3) & \frac{1}{\phi_3}(t(1-t^3)) & \frac{1}{\phi_3}(t^2(1+t+t^2)) \end{bmatrix} = \begin{bmatrix} a_3 & ta_1 & t^2a_1 \\ a_1 & ta_2 & t^2a_1 \\ a_1 & ta_1 & t^2 \end{bmatrix}.$$

Given the form of ζ^k is correct then we’ll examine ζ^{k+2} .

$$\zeta^{k+2} = \zeta^k * \zeta^2 = \begin{bmatrix} a_{\frac{3}{2}k} & ta_{\frac{3}{2}k-2} & t^2a_{\frac{3}{2}k-2} \\ a_{\frac{3}{2}k-2} & ta_{\frac{3}{2}k-1} & t^2a_{\frac{3}{2}k-2} \\ a_{\frac{3}{2}k-2} & ta_{\frac{3}{2}k-2} & t^2a_{\frac{3}{2}k-3} \end{bmatrix} \begin{bmatrix} a_3 & ta_1 & t^2a_1 \\ a_1 & ta_2 & t^2a_1 \\ a_1 & ta_1 & t^2 \end{bmatrix}.$$

Consider element $[1,1]$ of the matrix ζ^k , since k is even then $\frac{3}{2}k \equiv 0 \pmod{3}$, $\frac{3}{2}k-1 \equiv 2 \pmod{3}$ and $\frac{3}{2}k-3 \equiv 0 \pmod{3}$. Multiplying the first row and the first column we get the $[1,1]$ element of the matrix.

$$\begin{aligned}
[1, 1] &= a_{\frac{3}{2}k} * a_3 + ta_{\frac{3}{2}k-2} * a_1 + t^2 a_{\frac{3}{2}k-2} * a_1 \\
&= \frac{1}{\phi_3} (1 + t^{\frac{3}{2}k+1} + t^{\frac{3}{2}k+2}) (1 - t + t^3) + t * \frac{1}{\phi_3} (1 - t^{\frac{3}{2}k}) (1 - t) \\
&\quad + t^2 * \frac{1}{\phi_3} (1 - t^{\frac{3}{2}k}) * (1 - t) \\
&= \frac{1}{\phi_3} [(1 + t^{\frac{3}{2}k+1} + t^{\frac{3}{2}k+2}) - (t + t^{\frac{3}{2}k+2} + t^{\frac{3}{2}k+3}) + (t^3 + t^{\frac{3}{2}k+4} + t^{\frac{3}{2}k+5})] \\
&\quad + [t - t^{\frac{3}{2}k+1} - t^2 + t^{\frac{3}{2}k+2}] + [t^2 - t^{\frac{3}{2}k+2} - t^3 + t^{\frac{3}{2}k+3}] \\
&= \frac{1}{\phi_3} (1 + t^{\frac{3}{2}k+1} + t^{\frac{3}{2}k+2} - t - t^{\frac{3}{2}k+2} - t^{\frac{3}{2}k+3} + t^3 + t^{\frac{3}{2}k+4} + t^{\frac{3}{2}k+5} \\
&\quad + t - t^{\frac{3}{2}k+1} - t^2 + t^{\frac{3}{2}k+2} + t^2 - t^{\frac{3}{2}k+2} - t^3 + t^{\frac{3}{2}k+3}) \\
&= \frac{1}{\phi_3} (1 - t + t + t^2 - t^2 + t^3 - t^3 + t^{\frac{3}{2}k+1} - t^{\frac{3}{2}k+1} + t^{\frac{3}{2}k+2} - t^{\frac{3}{2}k+2} \\
&\quad + t^{\frac{3}{2}k+2} - t^{\frac{3}{2}k+2} + t^{\frac{3}{2}k+3} - t^{\frac{3}{2}k+3} + t^{\frac{3}{2}k+4} + t^{\frac{3}{2}k+5}) \\
&= \frac{1}{\phi_3} (1 + t^{\frac{3}{2}k+4} + t^{\frac{3}{2}k+5}) \\
&= a_{\frac{3}{2}k+3}, \text{ Note: We use the definition of } a_m. \\
&= a_{\frac{3}{2}(k+2)}
\end{aligned}$$

We see that the the general form for the ζ^k 's $[1,1]$ matrix element holds. Similar arguments can be used to prove each of the 9 elements of the even form. \square

3.3 Solution to Constraints

Lemma 2. *If $t = e^{\frac{2\pi i}{\tau}}$ then for any given $\tau \in \mathbb{N}$, such that $\tau \neq 3$, there exists $k \in \mathbb{N}$ such that $\zeta^k = I_3$.*

Proof. Let k be even and $t = e^{\frac{2\pi i}{\tau}}$. Then t is a τ^{th} root of unity. We will now solve for t given $\zeta^k = I_3$. Suppose the off-diagonal elements are equal to zero, so $-\frac{1}{\phi_3} \prod_{d \in S} \phi_d = \frac{1-t^{\frac{3k}{2}}}{\phi_3} = 0$. After redistributing terms, we see $t^{\frac{3k}{2}} = 1$ as long as t

is not evaluated at the third root of unity. Given $t^{\frac{3k}{2}} = 1$, consider the diagonal elements of the even form. Since k is even then $\frac{3}{2}k \equiv 0 \pmod{3}$, $\frac{3}{2}k - 1 \equiv 2 \pmod{3}$ and $\frac{3}{2}k - 3 \equiv 0 \pmod{3}$. Therefore the main diagonal entries are all equal to 1:

$$\begin{aligned}
a_{\frac{3}{2}k} &= \frac{1+t^{\frac{3}{2}k+1}+t^{\frac{3}{2}k+2}}{\phi_3} = \frac{1+(t)+(t^2)}{\phi_3} = \frac{1+t+t^2}{\phi_3} \\
&= 1. \\
ta_{\frac{3}{2}k-1} &= t * \frac{1+t^{\frac{3}{2}k-1}+t^{\frac{3}{2}k+1}}{\phi_3} = t * \frac{1+(t^{-1})+(t)}{\phi_3} = \frac{1+t+t^2}{\phi_3} \\
&= 1. \\
t^2 a_{\frac{3}{2}k-3} &= t^2 * \frac{1+t^{\frac{3}{2}k-2}+t^{\frac{3}{2}k-1}}{\phi_3} = t^2 * \frac{1+t^{-1}+t^2}{\phi_3} = \frac{1+t+t^2}{\phi_3} \\
&= 1.
\end{aligned}$$

By hypothesis $t = e^{\frac{2\pi i}{\tau}}$ this implies that $t^\tau = 1$. Which in turn imposes the constraint $t^{\frac{3}{2}k} = 1$ when $\frac{3}{2}k = j * \tau$ such that $j \in \mathbb{N}$ (i.e. $\frac{3}{2}k$ is a multiple of τ). Thus k must be divisible by 2 and τ . This is satisfied if $k = 2\tau$. Therefore given any τ there exists a k such that ζ^k is equal to the identity matrix. \square

3.4 Evaluated Burau Representation is Unfaithful

Theorem 1. *If the Burau representation is evaluated at any primitive root of unity, excepting the third, then the representation is unfaithful.*

Proof. In Lemma 2 we showed that if t is evaluated at any root of unity, excepting the third, then there exists a k such that $\zeta^k = I_3$. This implies the kernel of the Burau representation is non-trivial. Therefore the map is unfaithful. \square

4 Corollaries

4.1 Injectivity of the Burau Representation over all B_n

In the above proofs, we used the specific element $\sigma_1\sigma_2\sigma_1 \in B_4$. Using direct products and matrix segmentation, we can generalize our results to $\sigma_i\sigma_{i+1}\sigma_i \in B_n$ for all $n \geq 3$.

Corollary 1. *For all $n \geq 3$, the Burau representation of B_n evaluated all roots of unity excepting the third is unfaithful.*

Proof. We must consider that $\sigma_1\sigma_2\sigma_1$ does not exist in B_1 or B_2 . Thus we limit ourselves to $n \geq 3$. We can express the image of $\sigma_i\sigma_{i+1}\sigma_i \in B_n$ under the Burau representation, using a direct product, as the segmented matrix

$$\zeta = \left[\begin{array}{c|cc|c} I_{i-1} & 0 & 0 & \\ \hline 0 & \begin{array}{cc|c} 1-t & (1-t)t & t^2 \\ 1-t & t & 0 \\ 1 & 0 & 0 \end{array} & 0 & \\ \hline 0 & 0 & I_{n-(i+2)} & \end{array} \right].$$

When this matrix is expressed as a segmented matrix it is easy to see that the image of $(\sigma_i\sigma_{i+1}\sigma_i)^k$ under matrix multiplication is $I_{i-1} \oplus \begin{bmatrix} 1-t & (1-t)t & t^2 \\ 1-t & t & 0 \\ 1 & 0 & 0 \end{bmatrix}^k \oplus I_{n-(i+2)}$. As a result of

Lemma 1 and Lemma 2, for any given $\tau \in \mathbb{Z}$ such that $\tau \neq 3$, there exists a k such that $\zeta^k = I_n$. Therefore, the kernel of the evaluated Burau representation is non-trivial so the representation is unfaithful. \square

4.2 Relationship between root of unity and matrix power

Corollary 2. *For a given root of unity, τ , the the minimum power, k , that makes $(\sigma_1\sigma_2\sigma_1)^k$ equal to the identity matrix is $k = \frac{2}{3}\tau$ for $\tau \equiv 0 \pmod{3}$ and $k = 2\tau$ for $\tau \equiv 1$ or $2 \pmod{3}$.*

Proof. Given $t^{\frac{3k}{2}} = 1$, consider the diagonal elements of the even form. Since k is even $\frac{3}{2}k \equiv 0 \pmod{3}$, $\frac{3}{2}k - 1 \equiv 2 \pmod{3}$ and $\frac{3}{2}k - 3 \equiv 0 \pmod{3}$. Then the diagonal entries are:

$$\begin{aligned} a_{\frac{3}{2}k} &= \frac{1+t^{\frac{3}{2}k+1}+t^{\frac{3}{2}k+2}}{\phi_3} = \frac{(t^{\frac{3}{2}k})+t^{\frac{3}{2}k+1}+t^{\frac{3}{2}k+2}}{\phi_3} = t^{\frac{3}{2}k} * \frac{1+t+t^2}{\phi_3} \\ &= t^{\frac{3}{2}k}. \\ ta_{\frac{3}{2}k-1} &= t * \frac{1+t^{\frac{3}{2}k-1}+t^{\frac{3}{2}k+1}}{\phi_3} = t * \frac{(t^{\frac{3}{2}k-1})+t^{\frac{3}{2}k}+t^{\frac{3}{2}k+1}}{\phi_3} = t * t^{\frac{3}{2}k-1} * \frac{1+t+t^2}{\phi_3} \\ &= t^{\frac{3}{2}k}. \\ t^2a_{\frac{3}{2}k-3} &= t^2 * \frac{1+t^{\frac{3}{2}k-2}+t^{\frac{3}{2}k-1}}{\phi_3} = t^2 * \frac{(t^{\frac{3}{2}k-2})+t^{\frac{3}{2}k-1}+t^{\frac{3}{2}k}}{\phi_3} = t^2 * t^{\frac{3}{2}k-2} * \frac{1+t+t^2}{\phi_3} \\ &= t^{\frac{3}{2}k}. \end{aligned}$$

We see that the main diagonals are equal to $t^{\frac{3}{2}k}$, which, by hypothesis, equals 1. Furthermore, we let $t = e^{\frac{2\pi i}{\tau}}$ which implies that $t^\tau = 1$. Which in

turn imposes the constraint $t^{\frac{3}{2}k} = 1$ when $\frac{3}{2}k = j * \tau$ such that $j \in \mathbb{N}$ (i.e. $\frac{3}{2}k$ is a multiple of τ).

Since $\frac{3}{2}k$ is a multiple of 3, consider $\tau \bmod 3$.

1. $\tau = 3l$

Let $\tau = \frac{3}{2}k$ then $3l = \frac{3}{2}k$. Since both sides are divisible by 3, $l = \frac{k}{2}$. Therefore k must be divisible by 2 and l . The smallest term divisible by both 2 and l is $2l$. Note that $3l = \tau$, thus $\frac{2}{3}\tau = k$.

2. $\tau = 3l + 1$ or $\tau = 3l + 2$

Let $\tau = \frac{3}{2}k$ then $3l + 1 = \frac{3}{2}k$ or $3l + 2 = \frac{3}{2}k$. Neither value of τ is divisible by 3. Therefore k must be divisible by both 2 and τ . The smallest term divisible by both 2 and τ is 2τ , thus $2\tau = k$.

□

5 Future Work

It is the speculation of my advisor, Rebecca Winarski, that evaluating the Burau representation at a given root of unity denotes which covering space we are using for the disk of n marked points. (e.g. evaluating the representation at the 2nd root of unity corresponds to the hyperelliptic involution). This is only one case where the root of unity dictated the covering map, there is nothing proven that dictates a generalized relation between the root of unity and the covering map. The results of this work give information about the Braid relation when evaluated at the roots of unity, consequently it indicates what types of covering maps we should be looking at.

6 Conclusion

In this paper, we studied the braid group using representation theory. Given that we are evaluating the Burau representation at a root unity, excepting the third, we showed that the evaluated Burau representation of any braid group which holds the braid relation is unfaithful. This result arises from observing the periodicity of the cyclic groups generated by the image of $\sigma_1\sigma_2\sigma_2$ under the evaluated Burau representation. Furthermore, in corollary 2, we explicitly established the relationship between the root and the periodicity.

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7 Additional Material - NOT for Final Draft

7.1 Additional Topology

Definition 10. *Homotopy* - A continuous function $h : X \times [0, 1] \rightarrow Y$ such that if $x \in X$ then $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.

If we think of the second parameter of H as time then H describes a continuous deformation of f into g : at time 0 we have the function f and at time 1 we have the function g . We can also think of the second parameter as a *slider control* that allows us to smoothly transition from f to g as the slider moves from 0 to 1” - Wikipedia Homotopy

and let $p \in \partial D_n^\circ$. Let \tilde{p} denote the the full pre-image of p . As a $\mathbb{Z}[t, t^{-1}]$ -module, $H_1(Y_n, \tilde{p}; \mathbb{Z})$ has rank n ; the generators are represented by path lifts to Y_n of the loops γ_i in D_n° shown in the left-hand side of figure 1.”

Since the cover Y_n is characteristic, each element of B_n induces a t -equivariant homeomorphism of Y_n and the induced action is the Burau Representation.

Consider the relation $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ of B_n this relation is analogous to the third Reidemeister move for knots. It is my speculation that an explanation for why the 3rd root does not cause injectivity lies in that the element we’re using is connected to the third Reidemeister move

?and what the value of t implies about the lift from Dehn twists to the braid group generator. Just because this element is faithful does that mean the whole map is? Do we think that evaluating at the third root is unfaithful if so, why? if not, why?